then $\frac{x^2+N}{2x} = \frac{900+920}{60} = \frac{1820}{60} = \frac{91}{3} = 30\frac{1}{3}$ the second value of $\sqrt{920}$. Next make $\frac{91}{3} = \frac{n}{d}$; then $\frac{2n^2-1}{2dn} = \frac{2 \times 91^2 - 1}{2 \times 91 \times 3} = \frac{16561}{546} =$ 30·33150183, differing from the truth but by 6 in the tenth place of figures, the true number being 30·33150177.

And in this way may the square roots, in the table at the end of this volume, be easily found.

TRACT XXIV.

TO CONSTRUCT THE SQUARE AND CUBE ROOTS AND THE RECIPROCALS OF THE SERIES OF THE NATURAL NUMBERS.

1. *For the Square Roots.*

SINCE the square root of $a^2 + n$ is $a + \frac{n}{2a} - \frac{n^2}{8a^3} + \frac{n^3}{16a^5} -$ &c: therefore the series of the square roots of a^2, a^2+1, a^2+2, a^2+3, &c, and their 1st, 2d, 3d, 4th, &c differences, will be as below:

Nos.	Square Roots.	1st Diffs.	2d Diffs.	3d Diffs.
a^2	a			
a^2+1	$a + \frac{1}{2a} - \frac{1}{8a^3} + \frac{1}{16a^5}$	$\frac{1}{2a} - \frac{1}{8a^3} + \frac{1}{16a^5}$	$\frac{1}{4a^3} - \frac{3}{8a^5}$	$\frac{3}{8a^5}$
a^2+2	$a + \frac{2}{\cdot} - \frac{4}{\cdot} + \frac{8}{\cdot}$	$\frac{1}{\cdot} - \frac{3}{\cdot} + \frac{7}{\cdot}$	$\frac{1}{\cdot} - \frac{6}{\cdot}$	$\frac{3}{\cdot}$ &c.
a^2+3	$a + \frac{3}{\cdot} - \frac{9}{\cdot} + \frac{27}{\cdot}$	$\frac{1}{\cdot} - \frac{5}{\cdot} + \frac{19}{\cdot}$	$\frac{1}{\cdot} - \frac{9}{\cdot}$	
a^2+4	$a + \frac{4}{\cdot} - \frac{16}{\cdot} + \frac{64}{\cdot}$	$\frac{1}{\cdot} - \frac{7}{\cdot} + \frac{37}{\cdot}$		

Where, the columns of fractions having in each of them the same denominator, after the first line, in each class, a dot is written in the place of the denominators, to save the too frequent repetition of the same quantities. Now it is evident that, in every class, both of roots and of every set of differences, the first terms are all alike; and therefore, by the subtractions, it happens that every class of differences contains one term fewer than the one immediately preceding it. These differences are to be employed in constructing tables of square roots; and the extent to which the orders of differences are to be continued, must be regulated by the number of decimal figures to which the roots in the table are to be carried. In the above specimen the differences are continued as far as the 3d order, where the common first term is $\frac{s}{8a^5}$, which may be sufficiently small for constructing all the preceding orders of differences, and then the series of roots themselves, as far as to 7 places of decimals in each, when we commence with the number 1024, for the first square a^2, the root of which is 32. After this, the squares 1025, 1026, 1027, &c, continually increasing, their roots 32+, &c, proceed increasing also; but the series of numbers, in every order of differences, are all in a decreasing progression; so that the following orders are all found by taking each latter difference from the one immediately above it. Then, to construct the table of roots, having found the first term of each order of differences, as far as necessary, suppose to the 3d order; subtract that continually from the first of the 2d differences, which will complete the series of this order of differences. Then these being taken each from the first difference, the successive remainders will form the whole series of first differences.—Lastly, these first differences added continually with the first square root, will form the whole series of roots, from the first rational root, suppose 32, the root of the square number 1024, to be continued to the next rational root 33, or root of the next square number 1089. Then begin again, from this last square number, in like manner, with a new series of roots and differences, which are to be continued to the third square number 1156, the root of which is the next rational root 34. Then the like process is to be repeated again, and continued from the 3d to the 4th square number. And so on, continuing from each successive square number, to the next following one, as far as necessary; the last of each series of roots and differences always verifying the whole series from square to square.

The computation may begin at 1024, for the series of squares 1024, 1089, 1156, &c, their differences being 65, 67, 69, &c, and their roots 32, 33, 34, &c, as in the margin; in order to find the intermediate or irrational roots, to any proposed extent in decimals. The roots will be obtained true to different numbers of figures, according to the number of the orders of differences employed.

Roots.	Squares.	Diffs.
32	1024	
		65
33	1089	
		67
34	1156	
		69
35	1225	
		71
36	1296	

The first differences only will give the roots true to 5 places of figures, in commencing with the square 1024; the 2d differences will give the roots true to 9 places; the 3d differences to 12 places; and so on, as here below.

First, To find the Diffs.

$\frac{1}{2a} = 0{\cdot}015625$

$\frac{-1}{8a^3} = \ldots -38147$

$\frac{+1}{16a^5} = \ldots\ldots +18\tfrac{1}{2}$

1st dif. $0{\cdot}015621187$

$\frac{1}{4a^3} = \ 0{\cdot}000007629$

$\frac{-3}{8a^5} = \ldots\ldots -11$

2d dif. $0{\cdot}000007618$

$\frac{3}{8a^5} =$ 3d dif. . . 11

Then for the Roots.

3d Dif.	2d Difs.	1st Difs.	Roots.
·0⁷1 +	·00000762	·01562119	32·00000000
	761	·01561357	32·01562119
	760	·01560596	32·03123476
	758	·01559836	32·04684072
	757	·01559078	32·06243908
	756	·01558321	32·07802986
	756	·01557565	32·09361307
	754	·01556809	32·10918872
	753	·01556055	32·12475681
	752	·01555302	32·14031736
	750	·01554550	32·15587038
	750	·01553800	32·17141588
		·01553050	32·18695388

2. *For the Cube Roots.*

In the series and contrivances for constructing a table of cube roots of numbers, the process is exactly similar to that for the square roots, just above explained, in every respect, differing only in the terms of the general series by which the root of the binomial is expressed, viz, the series for $\sqrt[3]{(a^3+n)}$, instead of the series for $\sqrt{(a^2+n)}$. So that, all the explanation, and forms of process, being the same here, as in the former case, for the square roots, the repetition of these may here be dispensed with, and we shall only need to set down the series of roots and differences, with the calculation from them.

Now the general form of the series for $\sqrt[3]{(a^3+n)}$, or the cube root of a^3+n, is $a + \frac{n}{3a^2} - \frac{n^2}{9a^5} + \frac{5n^3}{81a^8} - \frac{10a^4}{243a^{11}}$ &c: therefore, expounding n by 1, 2, 3, &c, the series of the cube roots of a^3, a^3+1, a^3+2, a^3+3, &c, with their 1st, 2d, 3d, &c differences, will be as below:

Nos.	Cube Roots.	1st Diffs.	2d Diffs.	3d Diffs.
a^3	a	$\frac{1}{3a^2} - \frac{1}{9a^5} + \frac{5}{81a^8}$	$\frac{2}{9a^5} - \frac{10}{27a^8}$	$\frac{10}{27a^8}$
a^3+1	$a + \frac{1}{3a^2} - \frac{1}{9a^5} + \frac{5}{81a^8}$	$\frac{1}{\cdot} - \frac{3}{\cdot} + \frac{35}{\cdot}$	$\frac{2}{\cdot} - \frac{20}{\cdot}$	$\frac{10}{\cdot}$ &c.
a^3+2	$a + \frac{2}{\cdot} - \frac{4}{\cdot} + \frac{40}{\cdot}$	$\frac{1}{\cdot} - \frac{5}{\cdot} + \frac{95}{\cdot}$	$\frac{2}{\cdot} - \frac{30}{\cdot}$	
a^3+3	$a + \frac{3}{\cdot} - \frac{9}{\cdot} + \frac{135}{\cdot}$	$\frac{1}{\cdot} - \frac{7}{\cdot} + \frac{185}{\cdot}$		
a^3+4	$a + \frac{4}{\cdot} - \frac{16}{\cdot} + \frac{320}{\cdot}$			

Now here all the series converge faster than the like series for the square roots; because here the denominators, having higher powers, are larger than those in the former; consequently fewer terms will suffice in this case, than were requisite in the former, for an equal degree of accuracy, in all the differences and roots. The calculation for a few terms here follows.

First, To find the Diffs.

$\frac{1}{3a^2} = \cdot 0033333333$

$\frac{-1}{9a^5} = \ldots\ldots 11111$

$\frac{+5}{81a^8} = \ldots\ldots\ldots 6$

1st dif. $\cdot 0033322228$

$\frac{2}{9a^5} = \cdot 0000022222$

$\frac{-10}{27a^8} = \ldots\ldots\ldots 37$

2d dif. $\cdot 0000022185$

$\frac{10}{27a^8} =$ 3d dif. . . 37

Then for the Roots.

3d Dif.	2d Diffs.	1st Diffs.	Cube Roots.
$\cdot 0^8 37$	$\cdot 0^5 22185$	·0033322228	10·0000000000
	22148	33300043	0033322228
	22111	33277895	0066622271
	22074	33255784	0099900166
	22037	33233710	0133155950
	22000	33211673	0166389660
	21963	33189673	0199601333
	21926	33167710	0232791006
	21889	33145784	0265958716
	21852	33123895	0299104500
	21815	33102043	0332228395
	21778	33080228	0365330438
		33058450	0398410666
			0431469116

3. *For the Reciprocals of Numbers.*

The reciprocals of the natural numbers a, $a+1$, $a+2$, $a+3$, &c, are denoted by the fractions $\frac{1}{a}$, $\frac{1}{a+1}$, $\frac{1}{a+2}$, $\frac{1}{a+3}$, &c, where a is any integer number to commence with; which reciprocals, with their several orders of differences here follow.

Recips.	1st Diffs.	2d Diffs.	3d Diffs.
$\frac{1}{a}$			
$\frac{1}{a+1}$	$\frac{1}{a.a+1}$	$\frac{1.2}{a.a+1.a+2}$	$\frac{1.2.3}{a.a+1.a+2.a+3}$
$\frac{1}{a+2}$	$\frac{1}{a+1.a+2}$	$\frac{1.2}{a+1.a+2.a+3}$	$\frac{1.2.3}{a+1.a+2.a+3.a+4}$
$\frac{1}{a+3}$	$\frac{1}{a+2.a+3}$		

Here, if we would employ only the column of first differences, by actually multiplying the terms in their denominators, these, with their two orders of differences, will be as follow.

Where the first differences are in arithmetical progression, and the 2d differences equal, viz, the constant number 2. Hence the series of denominators will be very soon constructed, by two easy additions, the first of which is by the constant number 2. So, for instance, if a be $=1000$, then the first differences, and their denominators, will be thus: Where the column of first diffs. increases always by the number 2, and the column of denominators is constructed by adding the several first differences. These denominators are so large, that a very few figures in their quotients, will be sufficient to form, by one addition for each, the original column of reciprocals, to a great many places of figures. And these reciprocals will be verified and corrected at every 10th number; for any reciprocal whose denominator ends with a cipher, will have the same signifi-

Denoms.	1st Dif.	2d D
a^2+a		
a^2+3a+2	$2a+2$	2
a^2+5a+6	$2a+4$	2
$a^2+7a+12$	$2a+6$	

1st Dif.	Denoms.
2002	1001000
2004	1003002
2006	1005006

cant figures as the reciprocal of its 10th part, which, it is supposed, has been before found.

The above first differences and denominators will be sufficient to construct the table of reciprocals, commencing with the number 1000, as far as 9 places of decimals, the constant 2d difference being 2 in the 9th place, for a considerable way. Thus, dividing 1 by the several denominators above set down, gives for their quotients the annexed column of first diffs, and thence their annexed reciprocals, &c.

1st Diffs.	Reciprocals.	Nos.
·000000999	·000999001	1001
·000000997	·000998004	1002
·000000995	·000997009	1003
·000000993	·000996016	1004

But if a table of reciprocals be desired to a greater number of decimals, we might take in, and employ, the column of 2d differences also; by which means we should obtain the series of reciprocals to 12 places of decimals. And so on, for still more figures.

From the last two or three Tracts, may be constructed, or may be easily continued further, such tables as here next follow, of the reciprocals, squares, cubes, and roots of the natural series of integer numbers; the use of which is evidently to shorten the trouble of arithmetical calculations. The structure of the table is evident: the first column contains the natural series of numbers, from 1 to 1000; the 2d the squares of the same; the 3d the cubes; the 4th the reciprocals; the 5th the square roots; and lastly the cube roots of the same. The decimals, in the columns of reciprocals and roots, are all set down to the nearest figure in the last decimal place; that is, when the next figure, beyond the last place set down in the table, came out a 5 or more, the last figure was increased by 1; otherwise not; except in the repetends, which occurred among the reciprocals, where the real last figure is always set down. Those reciprocals which in the table have less than seven places of figures, are such as terminate, and are complete within that number, having nothing remaining; such as ·5 the reciprocal of 2, ·25 the reciprocal of 4, &c. The manner and cases of applying these numbers are generally evident: but it may be remarked, that the column of reciprocals (which are no other than the decimal values of the quotients, resulting from the division of unity, or 1, by each of the several numbers, from 1 to 1000), is not only useful in showing, by inspection, the quotient when the dividend is unity or 1, but is also applied with much advantage in changing many divisions into multiplications, whatever the dividend or numerators may be, which are much easier performed, being done by only multiplying the reciprocal of the divisor, as found in the table, by the dividend, for the quotient. It will also apply to good purpose in summing the terms of many converging series, as in the 8th of these Tracts, in which a few of the first terms, to be found by division, are taken out of this table, and then added together.