TRACT XXIII.

A NEW AND EASY METHOD FOR THE SQUARE ROOTS OF NUMBERS.—FROM MY MATHEMATICAL MISCEL. P. 323.

Problem.—Having given any nonquadrate number N; it is required to find a simple vulgar fraction $\frac{n}{d}$, the value of which shall be within any degree of nearness to $\sqrt{N}$, the surd root of N.

Investigation.—Since $\sqrt{N}$ is $=\frac{n}{d}$ nearly, or $d^2N=n^2$ nearly; let d^2N be $=n^2-D$. Then, since n, d, and N, are all integers by the supposition, D must also be an integer; and the smaller that integer is, the nearer will the value of $\frac{n}{d}$ be to $\sqrt{N}$, as is evident: therefore let $D = 1$ the smallest integer; then is $d^2N = n^2 - 1$, or $n^2 = d^2N + 1$: suppose this to be $= (dx-1)^2 = d^2x^2 - 2dx + 1$, where x is evidently some near value of $\sqrt{N}$; from this equation we have $d = \frac{2x}{x^2-N}$, and consequently $n = \sqrt{(d^2N-1)} = \frac{x^2+N}{x^2-N}$; hence theref. $\sqrt{N} = \frac{n}{d}$ is $= \frac{x^2+N}{2x}$ nearly.

Thus then the function $\frac{x^2+N}{2x}$ is an approximate value of $\sqrt{N}$, where x is to be assumed of any value whatever; but the nearer it is taken to $\sqrt{N}$, the nearer will the value of the fraction be to $\sqrt{N}$ required. And since $\frac{x^2+N}{2x}$ is always nearer to $\sqrt{N}$ than what x is, therefore assume any integer, or rational fraction, for x, but the nearer to $\sqrt{N}$ the more convenient, and write that assumed value of it in this expression, instead of it, so shall we have a nearer approximate rational value of $\sqrt{N}$; then use this last found value of $\sqrt{N}$ instead of x, in the same expression, and there will result a still nearer rational value of $\sqrt{N}$; and thus, by always substituting the last found value for x, in the fraction $\frac{x^2+N}{2x}$ or $\frac{1}{2}x + \frac{N}{2x}$, the result will be a still nearer value. And thus we may proceed to any degree of proximity required.

But a theorem somewhat easier for this continual substitution, may be thus raised: $\frac{n}{d}$ being any one approximate value of $\sqrt{N}$, write it instead of x, in the general function $\frac{x^2+N}{2x}$, then we have $\frac{n^2+Nd^2}{2dn}$ for the general approximation. That is, having assumed or found any one approximation $\frac{n}{d}$, the numerator of the next nearer approximation will be equal to the sum of the square of the numerator n and N times the square of the denominator of this one, and the denominator of the new one will be double the product of the numerator and denominator of this.

Or, a still easier continual approximation is $\frac{2n^2-1}{2dn} = \frac{n}{d} - \frac{1}{2dn}$, which is equal to the former, because n^2 is $= d^2N + 1$.

Example 1.—To find near rational values of the square root of the number 2.—Here $N = 2$. Take $1\frac{1}{2}$ or $\frac{3}{2}$ for the first value of x, as being nearly equal to $\sqrt{2}$. Then $n = 3$, and $d = 2$; therefore $\frac{2n^2-1}{2dn} = \frac{18-1}{12} = \frac{17}{12} = 1{\cdot}416$&c, for the next nearer value of $\sqrt{2}$. Again, take $\frac{17}{12} = \frac{n}{d}$; then $\frac{2n^2-1}{2dn} = \frac{2\times17^2-1}{2\times17\times12} = \frac{577}{408} = 1{\cdot}414215$, true for $\sqrt{2}$ to the last figure. And writing again $\frac{577}{408}$ for $\frac{n}{d}$, we obtain $\frac{665857}{470832} = 1{\cdot}414213562376$ for the value of $\sqrt{2}$, true to the last figure, which should be a 3, instead of a 6.

This small number is but an unfavourable example of the method, notwithstanding the ease and expedition with which the root has been so quickly obtained. For, the larger the given number N is, the quicker will the theorem approximate. Thus, taking for

Example 2.—To find the root of the number 920. Here $N = 920$, and $x = 30$ nearly. Now we must first use the rule $\frac{x^2+N}{2x}$, because x is taken $= 30$, below the true value. Hence then $\frac{x^2+N}{2x} = \frac{900+920}{60} = \frac{1820}{60} = \frac{91}{3} = 30\frac{1}{3}$ the second value of $\sqrt{920}$. Next make $\frac{91}{3} = \frac{n}{d}$; then $\frac{2n^2-1}{2dn} = \frac{2 \times 91^2 - 1}{2 \times 91 \times 3} = \frac{16561}{546} =$ 30·33150183, differing from the truth but by 6 in the tenth place of figures, the true number being 30·33150177.

And in this way may the square roots, in the table at the end of this volume, be easily found.

TRACT XXIV.

TO CONSTRUCT THE SQUARE AND CUBE ROOTS AND THE RECIPROCALS OF THE SERIES OF THE NATURAL NUMBERS.

1. *For the Square Roots.*

SINCE the square root of $a^2 + n$ is $a + \frac{n}{2a} - \frac{n^2}{8a^3} + \frac{n^3}{16a^5} -$ &c: therefore the series of the square roots of a^2, $a^2 + 1$, $a^2 + 2$, $a^2 + 3$, &c, and their 1st, 2d, 3d, 4th, &c differences, will be as below:

Nos.	Square Roots.	1st Diffs.	2d Diffs.	3d Diffs.
a^2	a	$\frac{1}{2a} - \frac{1}{8a^3} + \frac{1}{16a^5}$	$\frac{1}{4a^3} - \frac{3}{8a^5}$	$\frac{3}{8a^5}$ &c.
a^2+1	$a + \frac{1}{2a} - \frac{1}{8a^3} + \frac{1}{16a^5}$	$\frac{1}{\cdot} - \frac{3}{\cdot} + \frac{7}{\cdot}$	$\frac{1}{\cdot} - \frac{6}{\cdot}$	$\frac{3}{\cdot}$
a^2+2	$a + \frac{2}{\cdot} - \frac{4}{\cdot} + \frac{8}{\cdot}$	$\frac{1}{\cdot} - \frac{5}{\cdot} + \frac{19}{\cdot}$	$\frac{1}{\cdot} - \frac{9}{\cdot}$	
a^2+3	$a + \frac{3}{\cdot} - \frac{9}{\cdot} + \frac{27}{\cdot}$	$\frac{1}{\cdot} - \frac{7}{\cdot} + \frac{37}{\cdot}$		
a^2+4	$a + \frac{4}{\cdot} - \frac{16}{\cdot} + \frac{64}{\cdot}$			

Where, the columns of fractions having in each of them the same denominator, after the first line, in each class, a dot is written in the place of the denominators, to save the too frequent repetition of the same quantities. Now it is evident that, in every class, both of roots and of every set of differences, the first terms are all alike; and therefore, by the subtractions, it happens that every class of differences con-