which ingenious performance, it seems, was lost, for want of encouragement to publish it.

A small specimen of such numbers was published in the Philosophical Transactions for the year 1714, by Mr. Long of Oxford; but it was not till 1742 that a complete antilogarithmic canon was published by Mr. James Dodson, wherein he has computed the numbers corresponding to every logarithm from 1 to 100000, for 11 places of figures.

TRACT XXI.

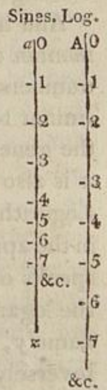
THE CONSTRUCTION OF LOGARITHMS, &c.

Having, in the last Tract, described the several kinds of logarithms, their rise and invention, their nature and properties, and given some account of the principal early cultivators of them, with the chief collections that have been published of such tables; proceed we now to deliver a more particular account of the ideas and methods employed by each author, and the peculiar modes of construction made use of by them. And first, of the great inventor himself, Lord Napier.

Napier's Construction of Logarithms.

The inventor of logarithms did not adapt them to the series of natural numbers 1, 2, 3, 4, 5, &c, as it was not his principal idea to extend them to all arithmetical operations in general; but he confined his labours to that circumstance which first suggested the necessity of the invention, and adapted his logarithms to the approximate numbers which express the natural sines of every minute in the quadrant, as they had been set down by former writers on trigonometry.

The same restricted idea was pursued through his method of constructing the logarithms. As the lines of the sines of all arcs are parts of the radius, or sine of the quadrant, which was therefore called the *sinus totus*, or whole sine, he conceived the line of the radius to be described, or run over, by a point moving along it in such a manner, that in equal portions of time it generated, or cut off, parts in a decreasing geometrical progression, leaving the several remainders, or sines, in geometrical progression also; while another point, in an indefinite line, described equal parts of *it* in the same equal portions of time; so that the respective sums of these, or the whole line generated, were always the arithmeticals or logarithms of these sines. Thus, *az* is the given radius on which all the sines are to be taken, and A&c the indefinite line containing the logarithms; these lines being each generated by the motion of points, beginning at A, *a*. Now, at the end of the 1st, 2d, 3d, &c, moments, or equal small portions of time, the moving points being found at the places marked 1, 2, 3, &c; then *za*, *z*1, *z*2, *z*3, &c, will be the series of natural sines, and A0, or 0, A1, A2, A3, &c, will be their logarithms; supposing the point which generates *az* to move every where with a velocity decreasing in proportion to its distance from *z*, namely, its velocity in the points 0, 1, 2, 3, &c, to be respectively as the distances *z*0, *z*1, *z*2, *z*3, &c, while the velocity of the point generating the logarithmic line A&c remains constantly the same as at first in the point A or 0.



Hitherto the author had not fully limited his system or scale of logarithms, having only supposed one condition or limitation, namely, that the logarithm of the radius *az* should be 0: whereas two independent conditions, no matter what, are necessary to limit the scale or system of logarithms. It did not occur to him that it was proper to form the other limit, by affixing some particular value to an assigned number, or part of the radius: but, as another condition was necessary, he assumed *this* for it, namely, that the two generating points should begin to move at *a* and A with equal velocities; or that the increments *a*1 and A1, described in the first moments, should be equal; as he thought this circumstance would be attended with some little ease in the computation. And this is the reason that, in his table, the natural sines and their logarithms, at the complete quadrant, have equal differences; and this is also the reason why his scale of logarithms happens accidentally to agree with what have since been called the hyperbolic logarithms, which have numeral differences equal to those of their natural numbers, at the beginning; except only that these latter increase with the natural numbers, and his on the contrary decrease; the logarithm of the ratio of 10 to 1 being the same in both, namely, 2·30258509.

And here, by the way, it may be observed, that Napier's manner of conceiving the generation of the lines of the natural numbers, and their logarithms, by the motion of points, is very similar to the manner in which Newton afterwards considered the generation of magnitudes in his doctrine of fluxions; and it is also remarkable, that, in art. 2, of the "Habitudines Logarithmorum et suorum naturalium numerorum invicem," in the appendix to the "Constructio Logarithmorum," Napier speaks of the velocities of the increments or decrements of the logarithms, in the same way as Newton does of his fluxions, namely, where he shows that those velocities, or fluxions, are inversely as the sines or natural numbers of the logarithms; which is a necessary consequence of the nature of the generation of those lines as described above; with this alteration, however, that now the radius *az* must be considered as generated by an equable motion of the point, and the indefinite line A&c by a motion increasing in the same ratio as the other before decreased; which is a supposition that Napier must have had in view when he stated that relation of the fluxions.

Having thus limited his system, Napier proceeds, in the posthumous work of 1619, to explain his construction of the logarithmic canon; and this he effects in various ways, but chiefly by generating, in a very easy manner, a series of proportional numbers, and their arithmeticals or logarithms; and then finding, by proportion, the logarithms to the natural sines, from those of the nearest numbers among the original proportionals.

After describing the necessary cautions he made use of, to preserve a sufficient degree of accuracy, in so long and complex a process of calculation; such as annexing several ciphers, as decimals separated by a point, to his primitive numbers, and rejecting the decimals thence resulting after the operations were completed; setting the numbers down to the nearest unit in the last figure; and teaching the arithmetical processes of adding, subtracting, multiplying, and dividing the limits, between which certain unknown numbers must lie, so as to obtain the limits between which the results must also fall; I say, after describing such particulars, in order to clear and smooth the way, he enters on the great field of calculation itself. Beginning at radius 10000000, he first constructs several descending geometrical series, but of such a nature, that they are all quickly formed by an easy continual subtraction, and a division by 2, or by 10, or 100, &c, which is done by only removing the decimal point so many places towards the left-hand, as there are ciphers in the divisor. He constructs three tables of such series: The first of these consists of 100 numbers, in the proportion of radius to radius minus 1, or of 10000000 to 9999999; all which are found by only subtracting from each its 10000000th part, which part is also found by only removing each figure seven places lower: the last of these 100 proportionals is found to be 9999900·0004950.

The 2d table contains 50 numbers, which are in the continual proportion of the first to the last in the first table, namely, of 10000000·0000000 to 9999900·0004950, or nearly the proportion of 100000 to 99999; these therefore are found by only removing the figures of each number 5 places lower, and subtracting them from the same number: the last of these he finds to be 9995001·222927. And a specimen of these two tables is here annexed.

No.	First Table.	Second Table.
1	10000000.0000000	10000000.000000
2	9999999.0000000	9999900.000000
3	9999998.0000001	9999800.001000
4	9999997.0000003	9999700.003000
&c.	&c till the 100th	&c to the 50th
50	term, which will be	term.
100	9999900.0004950	9995001.222927

The 3d table consists of 69 columns, and each column of 21 numbers or terms, which terms, in every column, are in the continual proportion of 10000 to 9995, that is, nearly as the first is to the last in the 2d table; and as 10000 exceeds 9995 by the 2000th part, the terms in every column will be constructed by dividing each upper number by 2, removing the figures of the quotient 3 places lower, and then subtracting them; and in this way it is proper to construct only the first column of 21 numbers, the last of which will be 9900473·5780: but the 1st, 2d, 3d, &c, numbers, in all the columns, are in the continual proportion of 100 to 99, or nearly the proportion of the first to the last in the first column; and therefore these will be found by removing the figures of each preceding number two places lower, and subtracting them, for the like number in the next column. A specimen of this 3d table is as here below.

The Third Table.

Terms	1st Column.	2d Column.	3d Column.	&c till the	69th Col.
1	10000000.0000	9900000.0000	9801000.0000	&c for	5048858.8900
2	9995000.0000	9895050.0000	9796099.5000	the 4th	5046334.4605
3	9990002.5000	9890102.4750	9791201.4503	5th, 6th,	5043811.2932
4	9985007.4987	9885157.4237	9786305.8495	7th, &c	5041289.3879
5	9980014.9950	9880214.8451	9781412.6967	col. till	5038768.7435
&c	&c till	&c	&c	the last	&c
21	9900473.5780	9801468.8423	9703454.1539	or	4998609.4034

Thus he had, in this 3d table, interposed between the radius and its half, 68 numbers in the continual proportion of 100 to 99; and interposed between every two of these, 20 numbers in the proportion of 10000 to 9995: and again, in the 2d table, between 10000000 and 9995000, the two first of the 3d table, he had 50 numbers in the proportion of 100000 to 99999; and lastly, in the 1st table, between 10000000 and 9999900, or the two first in the 2d table, 100 numbers in the proportion of 10000000 to 9999999; that is in all, about 1600 proportionals; all found in the most simple manner, by little more than easy subtractions; which proportionals nearly coincide with all the natural sines from 90° down to 30°.

To obtain the logarithms of all those proportionals, he demonstrates several properties and relations of the numbers and logarithms, and illustrates the manner of applying them. The principal of these properties are as follow: 1st, that the logarithm of any sine is greater than the difference between that sine and the radius, but less than the said difference when increased in the proportion of the sine to radius*; and 2dly, that the difference between the logarithms of two sines, is less than the difference of the sines increased in the proportion of the less sine to radius, but greater than the said difference of the sines increased in the proportion of the greater sine to radius †.

Hence, by the 1st theorem, the logarithm of 10000000, the radius or first term in the first table, being 0, the logarithm of 9999999, the 2d term, will be between 1 and 1·0000001, and will therefore be equal to 1·00000005 very nearly: and this will be also the common difference of all the terms or proportionals in the first table; therefore, by the continual addition of this logarithm, there will be obtained the logarithms of all these 100 proportionals; consequently 100 times the said first logarithm, or the last of the above sums, will

* By this first theorem, r being radius, the logarithm of the sine s is between $r-s$ and $\frac{r-s}{s}r$; and therefore, when s differs but little from r, the logarithm of s will be nearly equal to $\frac{(r+s)\times(r-s)}{2s}$, the arithmetical mean between the limits $r-s$ and $\frac{r-s}{s}r$; but still nearer to $(r-s)\sqrt{\frac{r}{s}}$ or $\frac{r-s}{s}\sqrt{rs}$, the geometrical mean between the said limits.

† By this second theorem, the difference between the logarithms of the two sines S and s, lying between the limits $\frac{S-s}{s}r$ and $\frac{S-s}{S}r$, will, when those sines differ but little, be nearly equal to $\frac{S^2-s^2}{2Ss}r$ or $\frac{(S+s)\times(S-s)}{2Ss}r$, their arithmetical mean; or nearly $\frac{S-s}{\sqrt{Ss}}r$, the geometrical mean; or nearly $=\frac{S-s}{S+s}2r$, by substituting in the last denominator, $\frac{1}{2}(S+s)$ for $\sqrt{Ss}$, to which it is nearly equal.

give 100·000005, for the logarithm of 9999900·0004950, the last of the said 100 proportions.

Then, by the 2d theorem, it easily appears, that ·0004950 is the difference between the logarithms of 9999900·0004950 and 9999900, the last term of the first table, and the 2d term of the second table; this then being added to the last logarithm, gives 100·0005000 for the logarithm of the said 2d term, as also the common difference of the logarithms of all the proportions in the second table; and therefore, by continually adding it, there will be generated the logarithms of all these proportionals in the second table; the last of which is 5000·025, answering to 9995001·222927, the last term of that table.

Again, by the 2d theorem, the difference between the logarithms of this last proportional of the second table, and the 2d term in the first column of the third table, is found to be 1·2235387; which being added to the last logarithm, gives 5001·2485387 for the logarithm of 9995000, the said 2d term of the third table, as also the common difference of the logarithms of all the proportionals in the first column of that table; and that this, therefore, being continually added, gives all the logarithms of that first column, the last of which is 100024·97077, the logarithm of 9900473·5780, the last term of the said column.

Finally, by the 2d theorem again, the difference between the logarithms of this last number and 9900000, the 1st term in the second column, is 478·3502; which being added to the last logarithm, gives 100503·3210 for the logarithm of the said 1st term in the second column, as well as the common difference of the logarithms of all the numbers on the same line in every line of the table, namely, of all the 1st terms, of all the 2d, of all the 3d, of all the 4th, &c, terms, in all the columns; and which, therefore, being continually added to the logarithms in the first column, will give the corresponding logarithms in all the other columns.

And thus is completed what the author calls the radical table, in which he retains only one decimal place in the logarithms (or *artificials*, as he always calls them in his tract on the construction), and four in the naturals. A specimen of the table is as here follows:

Radical Table.						
Terms	1st Column.		2d Column.		69th Column.	
	Naturals.	Artificals	Naturals.	Artific.	Naturals.	Artificials
1	10000000.0000	0	9900000.0000	100503.3	5048858.8900	6834225.8
2	9995000.0000	5001.2	9895050.0000	105504.6	5046333.4605	6839227.1
3	9990002.5000	10002.5	9890102.4750	110505.8	5043811.2932	6844228.3
4	9985007.4987	15003.7	9885157.4237	115507.1	5041289.3879	6849229.6
5	9980014.9950	20005.0	9880214.8451	120508.3	5038768.7435	6854230.8
&c	&c till	&c	&c	&c	&c	&c
21	9900473.5780	100025.0	9801468.8423	200528.2	4998609.4034	6934250.8

Having thus, in the most easy manner, completed the radical table, by little more than mere addition and subtraction, both for the natural numbers and logarithms; the logarithmic sines were easily deduced from it by means of the 2d theorem, namely, taking the sum and difference of each tabular sine and the nearest number in the radical table, annexing 7 ciphers to the difference, dividing the result by the sum, then half the quotient gives the difference between the logarithms of the said numbers, namely, between the tabular sine and radical number; consequently, adding or subtracting this difference, to or from the given logarithm of the radical number, there is obtained the logarithmic sine required. And thus the logarithms of all the sines, from radius to the half of it, or from 90° to 30°, were perfected.

Next, for determining the sines of the remaining 30 degrees, he delivers two methods. In the first of these he proceeds in this manner: Observing that the logarithm of the ratio of 2 to 1, or of half the radius, is 6931469·22, of 4 to 1 is the double of this, of 8 to 1 is triple of it, &c; that of 10 to 1 is 23025842.34, of 20 to 1 is the sum of the logarithms of 2 and 10; and so on, by composition for the logarithms of the ratios between 1 and 40, 80, 100, 200, &c, to 10000000; he multiplies any given sine, for an arc less than 30 degrees, by some of these numbers, till he finds the product nearly equal to one of the tabular numbers; then by means of this and the second theorem, the logarithm of this product is found; to which adding the logarithm that answers to the multiple above mentioned, the sum is the logarithm sought. But the other method is still much easier, and is derived from this property, which he demonstrates, namely, as half radius is to the sine of half an arc, so is the cosine of the said half arc, to the sine of the whole arc; or as $\frac{1}{2}$ radius : sine of an arc : : cosine of the arc : sine of double arc; hence the logarithmic sine of an arc is found, by adding together the logarithms of half radius and of the sine of the double arc, and then subtracting the logarithmic cosine from the sum.

And thus the remainder of the sines, from 30° down to 0, are easily obtained. But in this latter way, the logarithmic sines for full one half of the quadrant, or from 0 to 45 degrees, he observes, may be derived; the other half having already been made by the general method of the radical table, by one easy division and addition or subtraction for each.

We have dwelt the longer on this work of the inventor of logarithms, because I have not seen, in any author, an account of his method of constructing his table, though it is perfectly different from every other method used by the later computers, and indeed almost peculiar to his species of logarithms. The whole of this work manifests great ingenuity in the designer, as well as much accuracy. But notwithstanding the caution he took to obtain his logarithms true to the nearest unit in the last figure set down in the tables, by extending the numbers in the computations to several decimals, and other means; he had been disappointed of that end, either by the inaccuracy of his assistant computers or transcribers, or through some other cause; as the logarithms in the table are commonly very inaccurate. It is remarkable too, that in this tract on the construction of the logarithms, Lord Napier never calls them logarithms, but every where *artificials*, as opposed in idea to the natural numbers: and this notion, of natural and artificial numbers, I take to have been his first idea of this matter, and that he altered the word *artificials* to *logarithms* in his first book, on the description of them, when he printed it, in the year 1614, and that he would also have altered the word every where in this posthumous work, if he had lived to print it: for in the two or three pages of appendix, annexed to the work by his son, from Napier's papers, he again always calls them logarithms. This appendix relates to the change of the logarithms to that scale in which 1 is the logarithm of the ratio of 10 to 1, the logarithm of 1, with or without ciphers, being 0; and it appears to have been written after Briggs communicated to him his idea of that change.

Napier here in this appendix also briefly describes some methods, by which this new species of logarithms may be constructed. Having supposed 0 to be the logarithm of 1, and 1, with any number of ciphers, as 10000000000, the logarithm of 10; he directs to divide this logarithm of 10, and the successive quotients, ten times by 5; by which divisions there will be obtained these other ten logarithms, viz. 2000000000, 400000000, 80000000, 16000000, 3200000, 640000, 128000, 25600, 5120, 1024: then this last logarithm, and its quotients, being divided ten times by 2, will give these other ten logarithms, 512, 256, 128, 64, 32, 16, 8, 4, 2, 1. And the numbers answering to these twenty logarithms, we are directed to find in this manner; namely, extract the 5th root of 10, with ciphers, then the 5th root of that root, and so on, for ten continual extractions of the 5th root; so shall these ten roots be the natural numbers belonging to the first ten logarithms, above found in continually dividing by 5: next, out of the last 5th root we are to extract the square root, then the square root of this last root, and so on, for ten successive extractions of the square root; so shall these last ten roots be the natural numbers corresponding to the logarithms or quotients arising from the last ten divisions by the number 2. And from these twenty logarithms, 1, 2, 4, 8, 16, &c, and their natural numbers, the author observes that other logarithms and their numbers may be formed, namely, by adding the logarithms, and multiplying their corresponding numbers. It is evident that this process would generate rather an antilogarithmic canon, such as Dodson's, than the table of Briggs; and that the method would also be very laborious, since, besides the very troublesome original extractions of the 5th roots, all the numbers would be very large, by the multiplication of which the successive secondary natural numbers are to be found.

Our author next mentions another method of deriving a few of the primitive numbers and their logarithms, namely, by taking continually geometrical means, first between 10 and 1, then between 10 and this mean, and again between 10 and the last mean, and so on; and taking the arithmetical means between their corresponding logarithms. He then lays down various relations between numbers and their logarithms; such as, that the products and quotients of numbers answer to the sums and differences of their logarithms, and that the powers and roots of numbers answer to the products and quotients of the logarithms by the index of the power or root, &c; as also that, of any two numbers whose logarithms are given, if each number be raised to the power denoted by the logarithm of the other, the two results will be equal. He then delivers another method of making the logarithms to a few of the prime integer numbers, which is well adapted for constructing the common table of logarithms. This method easily follows from what has been said above; and it depends on this property, that the logarithm of any number in this scale, is 1 less than the number of places or figures contained in that power of the given number whose exponent is 10000000000, or the logarithm of 10, at least as to integer numbers, for they really differ by a fraction, as is shown by Mr. Briggs in his illustrations of these properties, printed at the end of this appendix to the construction of logarithms. We shall here just notice one more of these relations, as the manner in which it is expressed is exactly similar to that of fluxions and fluents, and it is this: Of any two numbers, as the greater is to the less, so is the velocity of the increment or decrement of the logarithms at the less, to the velocity of the increment or decrement of the logarithms at the greater: that is, in our modern notation, as $X : Y :: \dot{y}$ to $\dot{x}$, where $\dot{x}$ and $\dot{y}$ are the fluxions of the logarithms of X and Y.

Kepler's Construction of Logarithms.

The logarithms of Briggs and Kepler were both printed the same year, 1624; but as the latter are of the same sort as Napier's, we may first consider this author's construction of them, before proceeding to that of Briggs's.

We have already, in the last Tract, described the nature and form of Kepler's logarithms; showing that they are of the same kind as Napier's, but only a little varied in the form of the table. It may also be added, that, in general, the ideas which these two masters had on this subject, were of the same nature; only they were more fully and methodically laid down by Kepler, who expanded, and delivered in a regular science, the hints that were given by the illustrious inventor. The foundation and nature of their methods of construction are also the same, but only a little varied in their modes of applying them. Kepler here, first of any, treats of logarithms in the true and genuine way of the measures of ratios, or proportions*, as he calls them, and that in a very full and scientific manner: and this method of his was afterwards followed and abridged by Mercator, Halley, Cotes, and others, as we shall see in the proper places. Kepler first erects a regular and purely mathematical system of proportions, and the measures of proportions, treated at considerable length in a number of propositions, which are fully and chastely demonstrated by genuine mathematical reasoning, and illustrated by examples in numbers. This part contains and demonstrates both the nature and the principles of the struc-

* Kepler almost always uses the term *proportion* instead of *ratio*, which we shall also do in the account of his work, as well as conform in expressions and notations to his other peculiarities. It may also be here remarked, that I observe the same practice in describing the works of other authors, the better to convey the idea of their several methods and style. And this may serve to account for some seeming inequalities in the language of this history.

ture of logarithms. And in the second part the author applies those principles in the actual construction of his table, which contains only 1000 numbers, and their logarithms, in the form as we before described: and in this part he indicates the various contrivances made use of in deducing the logarithms of proportions one from another, after a few of the leading ones had been first formed, by the general and more remote principles. He uses the name *logarithms*, given them by the inventor, being the most proper, as expressing the very nature and essence of those artificial numbers, and containing as it were a definition in the very name of them; but without taking any notice of the inventor, or of the origin of those useful numbers.

As this tract is very curious and important in itself, and is besides very rare and little known, instead of a particular description only, we shall here give a brief translation of both the parts, omitting only the demonstrations of the propositions, and some rather long illustrations of them. The book is dedicated to Philip, landgrave of Hesse, but is without either preface or introduction, and commences immediately with the subject of the first part, which is intitled "The Demonstration of the Structure of Logarithms;" and the contents of it are as follow.

Postulate 1. That all proportions that are equal among themselves, by whatever variety of couplets of terms they may be denoted, are measured or expressed by the same quantity.

Axiom 1. If there be any number of quantities of the same kind, the proportion of the extremes is understood to be composed of all the proportions of every adjacent couplet of terms, from the first to the last.

1 *Proposition.* The mean proportional between two terms, divides the proportion of those terms into two equal proportions.

Axiom 2. Of any number of quantities regularly increasing, the means divide the proportion of the extremes into one proportion more than the number of the means.

Postulate 2. That the proportion between any two terms is divisible into any number of parts, until those parts become less than any proposed quantity.

An example of this section is then inserted in a small table, in dividing the proportion which is between 10 and 7 into 1073741824 equal parts, by as many mean proportionals wanting one, namely, by taking the mean proportional between 10 and 7, then the mean between 10 and this mean, and the mean between 10 and the last, and so on for 30 means, or 30 extractions of the square root, the last or 30th of which roots is 99999999966782056900; and the 30 power of 2, which is 1073741824, shows into how many parts the proportion between 10 and 7, or between 1000 &c, and 700 &c, is divided by 1073741824 means, each of which parts is equal to the proportion between 1000 &c, and the 30th mean 999&c, that is, the proportion between 1000&c, and 999&c, is the 1073741824th part of the proportion between 10 and 7. Then by assuming the small difference 00000000033217943100, for the measure of the very small element of the proportion of 10 to 7, or for the measure of the proportion of 1000&c, to 999&c, or for the logarithm of this last term, and multiplying it by 1073741824, the number of parts, the product gives 35667.49481.37222.14400, for the logarithm of the less term 7 or 700 &c.

Postulate 3. That the extremely small quantity or element of a proportion, may be measured or denoted by any quantity whatever; as for instance, by the difference of the terms of that element.

2 *Proposition.* Of three continued proportionals, the difference of the two first has to the difference of the two latter, the same proportion which the first term has to the 2d, or the 2d to the 3d.

3 *Prop.* Of any continued proportionals, the greatest terms have the greatest difference, and the least terms the least.

4 *Prop.* In any continued proportionals, if the difference of the greatest terms be made the measure of the proportion between *them*, the difference of any other couplet will be less than the true measure of *their* proportion.

5 *Prop.* In continued proportionals, if the difference of the greatest terms be made the measure of their proportion, then the measure of the proportion of the greatest to any other term will be greater than *their* difference.

6 *Prop.* In continued proportionals, if the difference of the greatest term and any one of the less, taken not immediately

next to it, be made the measure of their proportion, then the proportion which is between the greatest and any other term greater than the one before taken, will be less than the difference of those terms; but the proportion which is between the greatest term, and any one less than that first taken, will be greater than their difference.

7 *Prop.* Of any quantities placed according to the order of their magnitudes, if any two successive proportions be equal, the three successive terms which constitute them, will be continued proportionals.

8 *Prop.* Of any quantities placed in the order of their magnitudes, if the intermediates lying between any two terms be not among the mean proportionals which can be interposed between the said two terms, then such intermediates do not divide the proportion of those two terms into commensurable proportions.

Besides the demonstrations, as usual, several definitions are here given; as of commensurable proportions, &c.

9 *Prop.* When two expressible lengths are not to one another as two figurate numbers of the same species, such as two squares, or two cubes, there cannot fall between them other expressible lengths, which shall be mean proportionals, and as many in number as that species requires, namely, one in the squares, two in the cubes, three in the biquadrats, &c.

10 *Prop.* Of any expressible quantities, following in the order of their magnitudes, if the two extremes be not in the proportion of two square numbers, or two cubes, or two other powers of the same kind, none of the intermediates divide the proportion into commensurables.

11 *Prop.* All the proportions, taken in order, which are between expressible terms that are in arithmetical proportion, are incommensurable to one another. As between 8, 13, 18.

12 *Prop.* Of any quantities placed in the order of their magnitude, if the difference of the greatest terms be made the measure of their proportion, then the difference between any two others will be less than the measure of *their* propor-

tion; and if the difference of the two least terms be made the measure of their proportion, then the differences of the rest will be greater than the measure of the proportion between *their* terms.

Corol. If the measure of the proportion between the greatest exceed their difference, then the proportion of this measure to the said difference, will be less than that of a following measure to the difference of its terms. Because proportionals have the same ratio.

13 *Prop.* If three quantities follow one another in the order of magnitude, the proportion of the two least will be contained in the proportion of the extremes, a less number of times than the difference of the two least is contained in the difference of the extremes: And, on the contrary, the proportion of the two greatest will be contained in the proportion of the extremes, oftener than the difference of the former is contained in that of the latter.

Corol. Hence, if the difference of the two greater be equal to the difference of the two less terms, the proportion between the two greater will be less than the proportion between the two less.

14 *Prop.* Of three equidifferent quantities, taken in order, the proportion between the extremes is more than double the proportion between the two greater terms.

Corol. Hence it follows, that half the proportion of the extremes is greater than the proportion of the two greatest terms, but less than the proportion of the two least.

15 *Prop.* If two quantities constitute a proportion, and each quantity be lessened by half the greater, the remainders will constitute a proportion greater than double the former.

16 *Prop.* The aliquot parts of incommensurable proportions are incommensurable to each other.

17 *Prop.* If one thousand numbers follow one another in the natural order, beginning at 1000, and differing all by unity, viz. 1000, 999, 998, 997, &c; and the proportion between the two greatest 1000, 999, by continual bisection, be cut into parts that are smaller than the excess of the propor-

tion between the next two 999, 998, over the said proportion between the two greatest 1000, 999; and then for the measure of that small element of the proportion between 1000 and 999, there be taken the difference of 1000 and that mean proportional which is the other term of the element. Again, if the proportion between 1000 and 998 be likewise cut into double the number of parts which the former proportion, between 1000 and 999, was cut into; and then for the measure of the small element in this division, be taken the difference of its terms, of which the greater is 1000. And, in the same manner, if the proportion of 1000 to the following numbers, as 997, &c, by continual bisection, be cut into particles of such magnitude, as may be between $\frac{1}{2}$ and $\frac{3}{4}$ of the element arising from the section of the first proportion between 1000 and 999, the measure of each element will be given from the difference of its terms. Then, this being done, the measure of any one of the 1000 proportions will be composed of as many measures of its element, as there are of those elements in the said divided proportion. And all these measures, for all the proportions, will be sufficiently exact for the nicest calculations.

All these sections and measures of proportions are performed in the manner of that described at postulate 2, and the operation is abundantly explained by numerical calculations.

18 *Prop.* The proportion of any number, to the first term 1000, being known; there will also be known the proportion of the rest of the numbers in the same continued proportion, to the said first term.

So, from the known proportion between 1000 and 900, there is also known the prop. of 1000 to 810, and to 729;

And from 1000 to 800, also 1000 to 640, and to 512;
And from 1000 to 700, also 1000 to 490, and to 343;
And from 1000 to 600, also 1000 to 360, and to 216;
And from 1000 to 500, also 1000 to 250, and to 125.

Corol. Hence arises the precept for squaring, cubing, &c; as also for extracting the square root, cube root, &c. For it will be, as the greatest number of the chiliad, as a denominator, is to the number proposed as a numerator, so is this fraction to the square of it, and so is this square to the cube of it.

19 *Prop.* The proportion of a number to the first, or 1000, being known; if there be two other numbers in the same proportion to each other, then the proportion of one of these to 1000 being known, there will also be known the proportion of the other to the same 1000.

Corol. 1. Hence, from the 15 proportions mentioned in prop. 18, will be known 120 others below 1000, to the same 1000.

For so many are the proportions, equal to some one or other of the said 15, that are among the other integer numbers which are less than 1000.

Corol. 2. Hence arises the method of treating the Rule-of-Three, when 1000 is one of the given terms.

For this is effected by adding to, or subtracting from, each other, the measures of the two proportions of 1000 to each of the other two given numbers, according as 1000 is, or is not, the first term in the Rule-of-Three.

20 *Prop.* When four numbers are proportional, the first to the second as the third to the fourth, and the proportions of 1000 to each of the three former are known, there will also be known the proportion of 1000 to the fourth number.

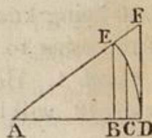
Corol. 1. By this means other chiliads are added to the former.

Corol. 2. Hence arises the method of performing the Rule-of-Three, when 1000 is not one of the terms. Namely from the sum of the measures of the proportions of 1000 to the second and third, take that of 1000 to the first, and the remainder is the measure of the proportion of 1000 to the fourth term.

Definition. The measure of the proportion between 1000 and any less number, as before described, and expressed by a number, is set opposite to that less number in the chiliad, and is called its LOGARITHM, that is, the number (αριθμος) indicating the proportion (λογον) which 1000 bears to that number, to which the logarithm is annexed.

21 *Prop.* If the first or greatest number be made the rad'c

of a circle, or *sinus totus;* every less number, considered as the cosine of some arc, has a logarithm greater than the versed sine of that arc, but less than the difference between the radius and secant of the arc; except only in the term next after the radius, or greatest term, the logarithm of which, by the hypothesis, is made equal to the versed sine.



That is, if CD be made the logarithm of AC, or the measure of the proportion of AC to AD; then the measure of the proportion of AB to AD, that is the logarithm of AB, will be greater than BD, but less than EF. And this is the same as Napier's first rule in page 345.

22 *Prop.* The same things being supposed; the sum of the versed sine and excess of the secant over the radius, is greater than double the logarithm of the cosine of an arc.

Corol. The log. cosine is less than the arithmetical mean between the versed sine and the excess of the secant.

Precept 1. Any sine being found in the canon of sines, and its defect below radius to the excess of the secant above radius, then shall the logarithm of the sine be less than half that sum, but greater than the said defect or coversed sine.

Let there be the sine	99970.1490	of an arc:
Its defect below radius is	29.8510	the covers. and less than the log. sine:
Add the excess of the secant	29.8599	
Sum	59.7109	
its half or	29.8555	greater than the logarithm.
Therefore the log. is between	29.8510	
and	29.8555	

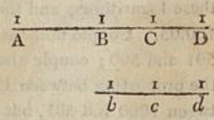
Precept 2. The logarithm of the sine being found, there will also be found nearly the logarithm of the round or integer number, which is next less than the sine with a fraction, by adding that fractional excess to the logarithm of the said sine.

Thus, the logarithm of the sine 99970.149 is found to be about 29.854; if now the logarithm of the round number 99970.000 be required, add 149, the fractional part of the sine, to its logarithm, observing the point, thus,

29.854
149

the sum 30.003 is the log. of the round number 99970.000 nearly.

23 *Prop.* Of three equidifferent quantities, the measure of the proportion between the two greater terms, with the measure of the proportion between the two less terms, will constitute a proportion, which will be greater than the proportion of the two greater terms, but less than the proportion of the two least.



Thus if AB, AC, AD be three quantities having the equal differences BC, CD; and if the measure of the proportion of AD, AC, *bc*, *cd*, and that of AC, AB be *bc*; then the proportion of *cd* to *cb* will be greater than the proportion of AC to AD, but less than the proportion of AB to AC.

24 *Prop.* The said proportion between the two measures is less than half the proportion between the extreme terms. That is, the proportion between *bc*, *cd*, is less than half the proportion between AB, AD.

Corol. Since therefore the arithmetical mean divides the proportion into unequal parts, of which the one is greater, and the other less, than half the whole; if it be inquired what proportion is between these proportions, the answer is, that it is a little less than the said half.

An Example of finding nearly the limits, greater and less, to the measure of any proposed proportion.

It being known that the measure of the proportion between 1000 and 900 is 10536.05, required the measure of the proportion 900 to 800, where the terms 1000, 900, 800, have equal differences. Therefore as 9 to 10, so 10536.05 to 11706.72, which is less than 11778.30 the measure of the proportion 9 to 8. Again, as the mean proportional between 8 and 10 (which is 8.9442719) is to 10, so 10536.05 to 11779.66, which is greater than the measure of the proportion between 9 and 8.

Axiom. Every number denotes an expressible quantity.

25 *Prop.* If the 1000 numbers, differing by 1, follow one another in the natural order; and there be taken any two adjacent numbers, as the terms of some proportion; the measure of this proportion will be to the measure of the proportion between the two greatest terms of the chiliad, in a proportion greater than that which the greatest term 1000 bears to the

greater of the two terms first taken, but less than the proportion of 1000 to the less of the said two selected terms.

So, of the 1000 numbers, taking any two successive terms, as 501 and 500, the logarithm of the former being 69114.92, and of the latter 69314.72, the difference of which is 199.80. Therefore, by the definition, the measure of the proportion between 501 and 500 is 199.80. In like manner, because the logarithm of the greatest term 1000 is 0, and of the next 999 is 100.05, the difference of these logarithms, and the measure of the proportion between 1000 and 999, is 100.05. Couple now the greatest term 1000 with each of the selected terms 501 and 500; couple also the measure 199.80 with the measure 100.05; so shall the proportion between 199.80 and 100.05, be greater than the proportion between 1000 and 501, but less than the proportion between 1000 and 500.

Corol. 1. Any number below the first 1000 being proposed, as also its logarithm, the differences of any logarithms antecedent to that proposed, towards the beginning of the chiliad, are to the first logarithm (viz. that which is assigned to 999) in a greater proportion than 1000 to the number proposed; but of those which follow towards the last logarithm, they are to the same in a less proportion.

Corol. 2. By this means, the places of the chiliad may easily be filled up, which have not yet had logarithms adapted to them by the former propositions.

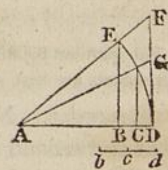
26 *Prop.* The difference of two logarithms, adapted to two adjacent numbers, is to the difference of these numbers, in a proportion greater than 1000 bears to the greater of those numbers, but less than that of 1000 to the less of the two numbers.

This 26th prop. is the same as Napier's second rule, at page 345.

27 *Prop.* Having given two adjacent numbers, of the 1000 natural numbers, with their logarithmic indices, or the measures of the proportions which those absolute or round numbers constitute with 1000, the greatest; the increments, or differences, of these logarithms, will be to the logarithm of the small element of the proportions, as the secants of the arcs whose cosines are the two absolute numbers, is to the greatest number, or the radius of the circle; so that, however, of the said two secants, the less will have to the radius a less proportion than the proposed difference has to the first of all,

but the greater will have a greater proportion, and so also will the mean proportional between the said secants have a greater proportion.

Thus if BC, CD be equal, also *bd* the logarithm of AB, and *cd* the logarithm of AC; then the proportion of *bc* to *cd* will be greater than the proportion of AG to AD, but less than that of AF to AD, and also less than that of the mean proportional between AF and AG to AD.



Corol. 1. The same obtains also when the two terms differ, not only by the unit of the small element, but by another unit, which may be ten fold, a hundred fold, or a thousand fold of that.

Corol. 2. Hence the differences will be obtained sufficiently exact, especially when the absolute numbers are pretty large, by taking the arithmetical mean between two small secants, or (if you will be at the labour) by taking the geometrical mean between two larger secants, and then by continually adding the differences, the logarithms will be produced.

Corol. 3. *Precept.* Divide the radius by each term of the assigned proportion, and the arithmetical mean (or still nearer the geometrical mean) between the quotients, will be the required increment; which being added to the logarithm of the greater term, will give the logarithm of the less term.

Example.

Let there be given the logarithm of 700, viz. 35667.4948, to find the log. to 699.

Here radius divided by 700 gives 1428571 &c.
and divided by 699 gives 1430672 &c.
the arithmetical mean is 142.962
which added to 35667.4948

gives the logarithm to 699   35810.4568

Corol. 4. *Precept* for the logarithms of sines.

The increment between the logarithms of two sines, is thus found: find the geometrical mean between the cosecants, and divide it by the difference of the sines, the quotient will be the difference of the logarithms.

Example.

0° 1′ sine 2909		cosec. 343774682
0 2 sine 5818		cosec. 171887319
dif. 2909,	geom. mean	2428 nearly.

The quotient 80000 exceeds the required increment of the logarithms, because the secants are here so large.

Appendix. Nearly in the same manner it may be shown, that the second differences are in the duplicate proportion of the first, and the third in the duplicate of the second. Thus, for instance, in the beginning of the logarithms, the first difference is 100.00000, viz. equal to the difference of the numbers 100000.00000 and 99900.00000; the second, or difference of the differences, 10000; the third 20. Again, after arriving at the number of 50000.00000, the logarithms have for a difference 200.00000, which is to the first difference, as the number 100000.00000 to 50000.00000; but the second difference is 40000, in which 10000 is contained 4 times; and the third 328, in which 20 is contained 16 times. But since in treating of new matters we labour under the want of proper words, therefore lest we should become too obscure, the demonstration is omitted untried.

28 *Prop.* No number expresses exactly the measure of the proportion, between two of the 1000 numbers, constituted by the foregoing method.

29 *Prop.* If the measures of all proportions be expressed by numbers or logarithms; all proportions will not have assigned to them their due portion of measure, to the utmost accuracy.

30 *Prop.* If to the number 1000, the greatest of the chiliad, be referred others that are greater than it, and the logarithm of 1000 be made 0, the logarithms belonging to those greater numbers will be negative.

This concludes the first or scientific part of the work, the principles of which Kepler applies, in the second part, to the actual construction of the first 1000 logarithms, which construction is pretty minutely described. This part is intitled

"A very compendious method of constructing the Chiliad of Logarithms;" and it is not improperly so called, the method being very concise and easy. The fundamental principles are briefly these: That at the beginning of the logarithms, their increments or differences are equal to those of the natural numbers: that the natural numbers may be considered as the decreasing cosines of increasing arcs: and that the secants of those arcs at the beginning have the same differences as the cosines, and therefore the same differences as the logarithms. Then, since the secants are the reciprocals of the cosines, by these principles and the third corollary to the 27th proposition, he establishes the following method of constituting the 100 first or smallest logarithms to the 100 largest numbers, 1000, 999, 998, 997, &c, to 900. viz. Divide the radius 1000, increased with seven ciphers, by each of these numbers separately, disposing the quotients in a table, and they will be the secants of those arcs which have the divisors for their cosines; continuing the division to the 8th figure, as it is in that place only that the arithmetical and geometrical means differ. Then by adding successively the arithmetical means between every two successive secants, the sums will be the series of logarithms. Or, by adding continually every two secants, the successive sums will be the series of the double logarithms.

Besides the 100 logarithms, thus constructed, the author constitutes two others by continual bisection, or extractions of the square root, after the manner described in the second postulate. And first he finds the logarithm which measures the proportion between 100000.00 and 97656.25, which latter term is the third proportional to 1024 and 1000, each with two ciphers; and this is effected by means of twenty-four continual extractions of the square root, determining the greatest term of each of twenty-four classes of mean proportionals; then the difference between the greatest of these means and the first or whole number 1000, with ciphers, being as often doubled, there arises 2371.6526 for the logarithm sought, which made negative is the logarithm of 1024. Secondly, the like process is repeated for the proportion between the numbers 1000 and 500, from which arises 69314.7193 for the logarithm of 500; which he also calls the logarithm of duplication, being the measure of the proportion of 2 to 1.

Then from the foregoing he derives all the other logarithms in the chiliad, beginning with those of the prime numbers 1, 2, 3, 5, 7, &c, in the first 100. And first, since 1024, 512, 256, 128, 64, 32, 16, 8, 4, 2, 1, are all in the continued proportion of 1000 to 500, therefore the proportion of 1024 to 1 is decuple of the proportion of 1000 to 500, and consequently the logarithm of 1 would be decuple of the logarithm of 500, if 0 were taken as the logarithm of 1024; but since the logarithm of 1024 is applied negatively, the logarithm of 1 must be diminished by as much: diminishing therefore 10 times the log. of 500, which is 693147.1928, by 2371.6526, the remainder 690775.5422 is the logarithm of 1, or of 100.00, which is set down in the table.

And because 1, 10, 100, 1000, are continued proportionals, therefore the proportion of 1000 to 1 is triple of the proportion of 1000 to 100, and consequently $\frac{1}{3}$ of the logarithm of 1 is to be set for the logarithm of 100, viz. 230258.5141, and this is also the logarithm of decuplication, or of the proportion of 10 to 1. And hence, multiplying this logarithm of 100 successively by 2, 3, 4, 5, 6, and 7, there arise the logarithms to the numbers in the decuple proportion, as in the margin.

Nos.	Logarithms.
100	230258.5141
10	460517.0282
1	690775.5422
.1	921034.0563
.01	1151292.5703
.001	1381551.0844
·0001	1611809.5985

Also if the logarithm of duplication, or of the proportion of 2 to 1, be taken from the logarithm of 1, there will remain the logarithm of 2; and from the logarithm of 2 taking the logarithm of 10, there remains the logarithm of the proportion of 5 to 1; which

Log. of 1	690775.5422
of 2 to 1	69314.7193
log. of 2	621460.8229
log. of 10	460517.0281
of 5 to 1	160943.7948
log. of 5	529831.7474

taken from the logarithm of 1, there remains the logarithm of 5. See the margin.

For the logarithms of other prime numbers he has recourse to those of some of the first or greatest century of numbers, before found, viz. of 999, 998, 997, &c. And first, taking 960, whose logarithm is 4082.2001; then by adding to this logarithm the logarithm of duplication, there will arise the several logarithms of all these numbers, which are in duplicate proportion continued from 960, namely 480, 240, 120, 60, 30, 15. Hence the logarithm of 30 taken from the logarithm of 10, leaves the logarithm of the proportion of 3 to 1; which taken from the logarithm of 1, leaves the logarithm of 3, viz. 580914.3106. And the double of this diminished by the logarithm of 1, gives 471053.0790 for the logarithm of 9.

Next, from the logarithm of 990, or 9 × 10 × 11, which is 1005.0331, he finds the logarithm of 11, namely, subtracting the sum of the logarithms of 9 and 10 from the sum of the logarithm of 990 and double the logarithm of 1, there remains 450986.0106 the logarithm of 11.

Again, from the logarithm of 980, or 2 × 10 × 7 × 7, which is 2020.2711, he finds 496184.5228 for the logarithm of 7.

And from 5129.3303 the logarithm of 950, or 5 × 10 × 19, he finds 396331.6392 for the logarithm of 19.

In like manner the logarithm

to 998 or 4 × 13 × 19, gives the logarithm of 13;
to 969 or 3 × 17 × 19, gives the logarithm of 17;
to 986 or 2 × 17 × 29, gives the logarithm of 29;
to 966 or 6 × 7 × 23, gives the logarithm of 23;
to 930 or 3 × 10 × 31, gives the logarithm of 31.

And so on for all the primes below 100, and for many of the primes in the other centuries up to 900. After which, he directs to find the logarithms of all numbers composed of these, by the proper addition and subtraction of their logarithms, namely, in finding the logarithm of the product of two numbers, from the sum of the logarithms of the two factors take the logarithm of 1, the remainder is the logarithm of the product. In this way he shows that the logarithms of all numbers under 500 may be derived, except those of the following 36 numbers, namely, 127, 149, 167, 173, 179, 211, 223, 251, 257, 263, 269, 271, 277, 281, 283, 293, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 397, 401, 409, 419, 421, 431, 433, 439, 443, 449. Also, besides the composite numbers between 500 and 900, made up of the products of some numbers whose logarithms have been before determined, there will be 59 primes not composed of them; which, with the 36 above mentioned, make 95 numbers in all not composed of the products of any before them, and the logarithms of which he directs to be derived in this manner; namely, by considering the differences of the logarithms of the numbers interspersed among them; then by that method by which were constituted the differences of the logarithms of the smallest 100 numbers in a continued series, we are to proceed here in the discontinued series, that is, by prop. 27, corol. 3, and especially by the appendix to it, if it be rightly used, whence those differences will be very easily supplied.

This closes the second part, or the actual construction of the logarithms; after which follows the table itself, which has been before described, pa. 323. Before dismissing Kepler's work however, it may not be improper in this place to take notice of an erroneous property laid down by him in the appendix to the 27th prop. just now referred to; both because it is an error in principle, tending to vitiate the practice, and because it serves to show that Kepler was not acquainted with the true nature of the orders of differences of the logarithms, notwithstanding what he says above with respect to the construction of them by means of their several orders of differences, and that consequently he has no legal claim to any share in the discovery of the differential method, known at that time to Briggs, and it would seem to him alone, it being published in his logarithms in the same year, 1624, as Kepler's book, together with the true nature of the logarithmic orders of differences, as we shall presently see in the following account of his works. Now this error of Kepler's, here alluded to, is in that expression where he says the third differences are in the *duplicate* ratio of the second differences, like as the second differences are in the duplicate ratio of the first; or, in other words, that the third differences are as the *squares* of the second differences, as well as the second differences as the squares of the first; or that the third differences are as the *fourth powers* of the first differences: Whereas in truth the third differences are only as the *cubes* of the first differences. Kepler seems to have been led into this error by a mistake in his numbers, viz. when he says in that appendix, that "*the third difference is* 328, *in which* 20 *is contained* 16 *times*;" for when the numbers are accurately computed, the third difference comes out only 161, in which therefore 20 is contained only 8 times, which is the cube of 2, the number of times the one first difference contains the other. It would hence seem that Kepler had hastily drawn the above erroneous principle from this one numerical example, or little more, false as it is: for had he made the trial in many instances, though erroneously computed, they could not easily have been so uniformly so, as to afford the same false conclusion in all cases. And therefore from hence, and what he says at the conclusion of that appendix, it may be inferred, that he either never attempted the demonstration of the property in question, or else that finding himself embarrassed with it, and unable to accomplish it, he therefore dispatched it in the ambiguous manner in which it appears.

But it may easily be shown, not only that the third differences of the logarithms at different places, are as the cubes of the first differences; but, in general, that the numbers in any one and the same order of differences, at different places, are as that power of the numbers in the first differences, whose index is the same as that of the order; or that the second, third, fourth, &c differences, are as the second, third, fourth, &c powers of the first differences. For the several orders of differences, when the absolute numbers differ by indefinitely small parts, are as the several orders of fluxions of the logarithms; but if x be any number, then $\frac{m\dot{x}}{x}$ is the fluxion of the logarithm of x, to the modulus m, and the second fluxion, or the fluxion of this fluxion, is $-\frac{m\dot{x}^2}{x^2}$, since $\dot{x}$ is constant; and the third, fourth, &c fluxions, are $\frac{2m\dot{x}^3}{x^3}$, $-\frac{2.3m\dot{x}^4}{x^4}$, &c; that is, the first, second, third, fourth, fifth, sixth, &c orders of fluxions, are equal to the modulus m multiplied into each of these terms,

$$\frac{\dot{x}}{x}, -\frac{1\dot{x}^2}{x^2}, \frac{1.2\dot{x}^3}{x^3}, -\frac{1.2.3\dot{x}^4}{x^4}, \frac{1.2.3.4\dot{x}^5}{x^5}, -\frac{1.2.3.4.5\dot{x}^6}{x^6}, \&c;$$

where it is evident, that the fluxion of any order is as that power of the first fluxion, whose index is the same as the number of the order. And these quantities would actually be the several terms of the differences themselves, if the differences of the numbers were indefinitely small. But they vary the more from them, as the differences of the absolute numbers differ from $\dot{x}$, or as the said constant numerical difference 1 approaches towards the value of x the number itself. However, on the whole, the several orders vary proportionably, so as still sensibly to preserve the same analogy, namely, that two nth differences are in proportion as the nth powers of their respective first differences.

Of Briggs's Construction of his Logarithms.

Nearly according to the methods described in p. 349, 350, Mr. Briggs constructed the logarithms of the prime numbers, as appears from his relation of this business in the "Arithmetica Logarithmica," printed in 1624, where he details, in an ample manner, the whole construction and use of his logarithms. The work is divided into 32 chapters or sections. In the first of these, logarithms in a general sense are defined, and some properties of them illustrated. In the second chapter he remarks, that it is most convenient to make 0 the logarithm of 1; and on that supposition he exemplifies these following properties, namely, that the logarithms of all numbers are either the indices of powers, or proportional to them; that the sum of the logarithms of two or more factors, is the logarithm of their product; and that the difference of the logarithms of two numbers, is the logarithm of their quotient. In the third section he states the other assumption, which is necessary to limit his system of logarithms, namely, making 1 the logarithm of 10, as that which produces the most convenient form of logarithms: He hence also takes occasion to show that the powers of 10, namely 100, 1000, &c, are the only numbers which can have rational logarithms. The fourth section treats of the characteristic; by which name he distinguishes the integral, or first part, of a logarithm towards the left hand, which expresses one less than the number of integer places or figures, in the number belonging to that logarithm, or how far the first figure of this number is removed from the place of units; namely, that 0 is the characteristic of the logarithms of all numbers from 1 to 10; and 1 the characteristic of all those from 10 to 100; and 2 that of those from 100 to 1000; and so on.

He begins the fifth chapter with remarking, that his logarithms may chiefly be constructed by the two methods which were mentioned by Napier, as above related, and for the sake of which, he here premises several *lemmata*, concerning the powers of numbers and their indices, and how many places of figures are in the products of numbers, observing that the product of two numbers will consist of as many figures as there are in both factors, unless perhaps the product of the first figures in each factor be expressed by one figure only, which often happens, and then commonly there will be one figure in the product less than in the two factors; as also that, of any two of the terms, in a series of geometricals, the results will be equal by raising each term to the power denoted by the index of the other; or any number raised to the power denoted by the logarithm of the other, will be equal to this latter number raised to the power denoted by the logarithm of the former; and consequently if the one number be 10, whose logarithm is 1 with any number of ciphers, then any number raised to the power whose index is 1000 &c, or the logarithm of 10, will be equal to 10 raised to the power whose index is the logarithm of that number; that is, the logarithm of any number in this scale, where 1 is the logarithm of 10, is the index of that power of 10 which is equal to the given number. But the index of any integral power of 10, is one less than the number of places in that power; consequently the logarithm of any other number, which is no integral power of 10, is not quite one less than the number of places in that power of the given number whose index is 1000 &c, or the logarithm of 10.

Find therefore the 10th, or 100th, or 1000th &c, power of any number, as suppose 2, with the number of figures in such power; then shall that number of figures always exceed the logarithm of 2, though the excess will be constantly less than 1.

An example of this process is here given in the margin; where the 1st column contains the several powers of 2, the 2d their corresponding indices, and the 3d contains the number of places in the powers in the first column; and of these numbers in the third column, such as are on the lines of those indices that consist of 1 with ciphers, are continual approximations to the logarithm of 2, being always too great by less than 1 in the last figure, that logarithm being 30102999566398 &c.

And here, since the exact powers of 2 are not required, but only the number of figures they consist of, as shown by the third column, only a few of the first figures of the powers in the first column are retained, those being sufficient to determine the num-

Powers of 2	Indices.	No. of Places or logs.
2	1	1
4	2	1
16	4	2
256	8	3
1024	10	4 log. of 2
10486	20	7 log. of 4
10995	40	13 log. of 16
12089	80	25 log. of 256
12676	100	31 log. of 2
16069	200	61 log. of 4
25823	400	121 log. 16
66680	800	241 log. 256
10715	1000	302 log. 2
11481	2000	603 log. 4
13182	4000	1205 log. 16
17377	8000	2409 log. 256
19950	10000	3011 log. 2
39803	20000	6021 log. 4
15843	40000	12042 log. 16
25099	80000	24083 log. 256
99900	100000	30103 log. 2
99801	200000	60206 log. 4
99601	400000	120412 &c.
99204	800000	240824
99006	1000000	301030
98023	2000000	602060
96085	4000000	1204120
92323	8000000	2408240
90498	10000000	3010300
81899	20000000	6020600
67075	40000000	12041200
44990	80000000	24082400
36846	100000000	30103000
13577	200000000	60206000
18433	400000000	120411999
33977	800000000	240823997
46129	1000000000	301029996

ber of places in them; and the multiplications in raising these powers are performed in a contracted way, so as to have the fifth or last figure in them true to the nearest unit. Indeed these multiplications might be performed in the same manner, retaining only the first three figures, and those to the nearest unit in the third place; which would make this a very easy way indeed of finding the logarithms of a few prime numbers.

It may also be remarked, that those several powers, whose indices are 1 with ciphers, are raised by thrice squaring from the former powers, and multiplying the first by the third of these squares; making also the corresponding doublings and additions of their indices: thus, the square of 2 is 4, and the square of 4 is 16, the square of 16 is 256, and 256 multiplied by 4 is 1024; in like manner, the double of 1 is 2, the double of 2 is 4, the double of 4 is 8, and 8 added to 2 makes 10. And the same for all the following powers and indices. The numbers in the third column, which show how many places are in the corresponding powers in the first column, are produced in the very same way as those in the second column, namely, by three duplications and one addition; only observing to subtract 1 when the product of the first figures are expressed by one figure; or when the first figures exceed those of the number or power next above them. It may further be observed, that, like as the first number in each quaternion, or space of four lines or numbers, in the third column, approximates to the logarithm of 2, the first number in the first quaternion of the first column; so the second, third, and fourth terms of each quaternion in the third column, approximate to the logarithm of 4, 16, and 256, the second, third, and fourth numbers in the first quaternion in the first column. And further, by cutting off one, two, three, &c, figures, as the index or integral part, from the said logarithms of 2, 4, 16, and 256, the first, second, third, and fourth numbers in the first quaternion of the first column, the remaining figures will be the decimal part of the logarithms of the corresponding first, second, third, and fourth numbers in the following second, third, fourth, &c,

quaternions: the reason of which is, that any number of any quaternion in the first column, is the tenth power of the corresponding term in the next preceding quaternion. So that the third column contains the logarithms of all the numbers in the first column: a property which, if Dr. Newton had been aware of, he could not easily have committed such gross mistakes as are found in a table of his, similar to that above given, in which most of the numbers in the latter quaternions are totally erroneous; and his confused and imperfect account of this method would induce one to believe that he did not well understand it.

In the sixth chapter our illustrious author begins to treat of the other general method of finding the logarithms of prime numbers, which he thinks is an easier way than the former, at least when the logarithm is required to a great many places of figures. This method consists in taking a great number of continued geometrical means between 1 and the given number whose logarithm is required; that is, first extracting the square root of the given number, then the root of the first root, the root of the second root, the root of the third root, and so on till the last root shall exceed 1 by a very small decimal, greater or less according to the intended number of places to be in the logarithm sought: then finding the logarithm of this small number, by methods described below, he doubles it as often as he made extractions of the square root, or, which is the same thing, he multiplies it by such power of 2 as is denoted by the said number of extractions, and the result is the required logarithm of the given number; as is evident from the nature of logarithms. The rule to know how far to continue this extraction of roots is, that the number of decimal places in the last root, be double the number of true places required to be found in the logarithm, and that the first half of them be ciphers; the integer being 1: the reason of which is, that then the significant figures in the decimal, after the ciphers, are directly proportional to those in the corresponding logarithms; such figures in the natural number being the half of those in the next preceding number, like as the logarithm of the last number is the half of the preceding logarithm. Therefore, any one such small number, with its logarithm, being once found, by the continual extractions of square roots out of a given number, as 10, and corresponding bisections of its given logarithm 1; the logarithm for any other such small number, derived by like continual extractions from another given number, whose logarithm is sought, will be found by one single proportion: which logarithm is then to be doubled according to the number of extractions, or multiplied at once by the like power of 2, for the logarithm of the number proposed. To find the first small number and its logarithm, our author begins with the number 10 and its logarithm 1, and extracts continually the root of the last number, and bisects its logarithm, as here registered in the margin, but to far more places of figures, till he arrives at the 53d and 54th roots, with their annexed logarithms, as here below:

	10, given n°.	1, its log.
1	3·162277 &c	0·5
2	1·778279	0·25
3	1·333521	0·125
4	1·154781	0·0625
5	1·074607	0·03125
	&c.	&c.

	Numbers.	Logarithms.
35	1·00006,00000,00000,25563,82986,40064,70	0·00000,00000,00000,11102,23024,62515,65404
54	1·00000,00000,00000,12781,91493,20032,35	0·00000,00000,00000,05551,11512,31257,82702

where the decimals in the natural numbers are to each other in the ratio of the logarithms, namely in the ratio of 2 to 1: and therefore any other such small number being found, by continual extraction or otherwise, it will then be as 12781&c, is to 5551&c, so is that other small decimal, to the corresponding significant figures of its logarithm. But as every repetition of this proportion requires both a very long multiplication and division, he reduces this constant ratio to another equivalent ratio whose antecedent is 1, by which all the divisions are saved: thus,

as 12781&c : 5551&c : : 1000&c : 434294481903251804,
that is, the logarithm of 1·00000,00000,00000,1
is 0·00000,00000,00000,04342,94481,90325,1804;

and therefore this last number being multiplied by any such small decimal, found as above by continual extraction, the product will be the corresponding logarithm of such last root.

But as the extraction of so many roots is a very troublesome operation, our author devises some ingenious contrivances to abridge that labour. And first, in the 7th chapter, by the following device, to have fewer and easier extractions to perform: namely, raising the powers from any given prime number, whose logarithm is sought, till a power of it be found such that its first figure on the left hand is 1, and the next to it either one or more ciphers; then, having divided this power by 1 with as many ciphers as it has figures after the first, or supposing all after the first to be decimals, the continual roots from this power are extracted till the decimal become sufficiently small, as when the first fifteen places are ciphers; and then by multiplying the decimal by 43429 &c, he has the logarithm of this last root; which logarithm multiplied by the like power of the number 2, gives the logarithm of the first number, from which the extraction was begun: to this logarithm prefixing a 1, or 2, or 3, &c, according as this number was found by dividing the power of the given prime number by 10, or 100, or 1000, &c; and lastly, dividing the result by the index of that power, the quotient will be the required logarithm of the given prime number. Thus, to find the logarithm of 2: it is first raised to the 10th power, as in the margin, before the first figures come to be 10; then, dividing by 1000, or cutting off for decimals all the figures after the first or 1, the root is continually extracted out of the quotient 1,024, till the 47th extraction, which gives 1.00000,00000,00000,16851,60570,53949,77; the decimal part of which multi. by 43429 &c, gives 0.00000,00000,00000,07318,55936,90623,9368 for its logarithm: and this being continually doubled for 47 times, gives the logarithms of all the roots up to the first number: or being at once

2	1
4	2
8	3
16	4
32	5
64	6
128	7
256	8
512	9
1024	10

multiplied by the 47th power of 2, viz. 140737488355328, which is raised as in the margin, it gives 0·01029,99566,39811,95265,27744 for the logarithm of the number 1.024, true to 17 or 18 decimals: to this prefix 3, so shall 3.0102 &c be the logarithm of 1024: and lastly, because 2 is the tenth root of 1024, divide by 10, so shall 0.30102,99956,63981,1952 be the logarithm required to the given number 2.

2	1
4	2
8	3
16	4
32	5
64	6
128	7
256	8
512	9
1024	10
1048576	20
1073741824	30
1099511627776	40
140737488355328	47

The logarithms of 1, 2, and 10 being now known; it is remarked that the logarithm of 5 becomes known; for since 10 ÷ 2 is = 5, therefore log. 10 − log. 2 = log. 5, which is 0.69897,00043,36018,8058; and that from the multiplications and divisions of these three 2, 5, 10, with the corresponding additions and subtractions of their logarithms, a multitude of other numbers and their logarithms are produced; so, from the powers of 2, are obtained 4, 8, 16, 32, 64, &c; from the powers of 5, these, 25, 125, 625, 3125, &c; also the powers of 5 by those of 10 give 250, 1250, 6250, &c; and the powers of 2 by those of 10, give 20, 200, 2000, &c; 40, 400, 80, 800, &c; likewise by division are obtained $2\frac{1}{2}$, $1\frac{1}{4}$, $12\frac{1}{2}$, $6\frac{1}{4}$, $1\frac{3}{5}$, $3\frac{1}{5}$, $6\frac{2}{5}$, &c.

Briggs then observes, that the logarithm of 3, the next prime number, will be best derived from that of 6, in this manner: 6 raised to the 9th power becomes 10077696, which divided by 10000000, gives 1.0077696, and the root from this continually extracted till the 46th,
is 1,00000,00000,00000,10998,59345,88155,71866;
the decimal part of which multiplied by 43429&c, gives
0.00000,00000,00000,04776,62844,78608,0304 for its logarithm; and this 46 times doubled, or multiplied by the 46th power of 2, gives 0.00336,12534,52792,69 for the logarithm

of 1.0077696; to which adding 7, the logarithm of the divisor 10000000, and dividing by 9, the index of the power of 6, there results 0.77815,12503,83643,63 for the logarithm of 6; from which subtracting the logarithm of 2, there remains 0.47712,12547,19662,44 for the logarithm of 3.

In the eighth chapter our ingenious author decribes an original and easy method of constructing, by means of differences, the continual mean proportionals which were before found by the extraction of roots. And this, with the other methods of generating logarithms by differences, in this book as well as in his "Trigonometria Britannica," are I believe the first instances that are to be found of making such use of differences, and show that he was the inventor of what may be called the "Differential Method." He seems to have discovered this method in the following manner: having observed that these continual means between 1 and any number proposed, found by the continual extraction of roots, approach always nearer and nearer to the halves of each preceding root, as is visible when they are placed together under each other; and indeed it is found that as many of the significant figures of each decimal part, as there are ciphers between them and the integer 1, agree with the half of those above them; I say, having observed this evident approximation, he subtracted each of these decimal parts, which he called A, or the first differences, from half the next preceding one, and by comparing together the remainders or second differences, called B, he found that the succeeding were always nearly equal to $\frac{1}{4}$ of the next preceding ones; then taking the difference between each second difference and $\frac{1}{4}$ of the preceding one, he found that these third differences, called C, were nearly in the continual ratio of 8 to 1; again taking the difference between each C and $\frac{1}{8}$ of the next preceding, he found that these fourth differences, called D, were nearly in the continual ratio of 16 to 1; and so on, the 5th E, 6th F, &c, differences, being nearly in the continual ratio of 32 to 1, of 64 to 1, &c.

These plain observations being made, they very naturally and clearly suggested to him the notion and method of constructing all the remaining numbers, from the differences of a few of the first, found by extracting the roots in the usual way. This will evidently appear from the annexed specimen of a few of the first numbers in the last example, for finding the logarithm of 6; where, after the 9th number, the rest are supposed to be constructed from the preceding differences of each, as here shown in the 10th and 11th. And it is evident that, in proceeding, the trouble will become always less and less, the differences gradually vanishing, till at last only the first differences remain; and that generally each less difference is shorter than the next greater, by as many

	1,00776,96	
1	1,00387,72833,36962,45663,84655,1	
2	1,00193,67661,36946,61675,87022,9	
3	1,00096,79146,39099,01728,89072,0	
4	1,00048,38402,68846,62985,49253,5	A
5	1,00024,18908,78824,68563,80872,7	A
	24,19201,34423,31492,74626,7	$\frac{1}{2}$A
	292,55598,62928,93754,0	B
6	1,00012,09381,26397,13459,43919,4	A
	12,09454,39412,34281,90436,3	$\frac{1}{2}$A
	73,13015,20822,46516,9	B
	73,13899,65732.23438,5	$\frac{1}{4}$B
	884,44909,76921,5	C
7	1,00006,04672,35055,30968,01600,5	A
	6,04690.63198,56729,71959,7	$\frac{1}{2}$A
	18,28143,25761,70359,2	B
	18,28253,80205,61629,2	$\frac{1}{4}$B
	110,54443,91270,0	C
	110,55613,72115,2	$\frac{1}{8}$C
	1169.80845,2	D
8	1,00003,02331,60505,65775,96479,4	A
	3,02336,17527,65484,00800,2	$\frac{1}{2}$A
	4,57021,99708,04320,8	B
	4,57035,81440,42589,8	$\frac{1}{4}$B
	13,81732,38269,0	C
	13,81805,48908,7	$\frac{1}{8}$C
	73,10639,7	D
	73,11302,8	$\frac{1}{16}$D
	663,1	E
9	1,00001,51164,65999,05672,95048,8	A
	1,51165,80252,82887,98239,7	$\frac{1}{2}$A
	1,14253,77215,03190,9	B
	Hitherto the 1,14255,49927,01080,2	$\frac{1}{4}$B
	smaller differences 1,72711,97889,3	C
	are found by sub- 1,72716,54783,6	$\frac{1}{8}$C
	tracting the larger from 4,56894,3	D
	the parts of the like pre- 4,56915,0	$\frac{1}{16}$D
	ceding ones. 20,7	E
	20,7	$\frac{1}{32}$E
	Here the greater differences 65	$\frac{1}{32}$E
	remain after subtracting 28555,89	$\frac{1}{16}$D
	the smaller from the parts 28555,24	D
	of the difference of 21588,99736,16	$\frac{1}{8}$C
	the next preceding 21588,71180,92	C
	number. 28563,44303,75797,72	$\frac{1}{4}$B
	28563,22715,04616,80	B
	75582,32999,52836,47524,40	$\frac{1}{2}$A
10	1,00000,75582,04436,30121,42907,60	A
	2	$\frac{1}{32}$E
	1784,70	$\frac{1}{16}$D
	1784,68	D
	2693,58897,62	$\frac{1}{8}$C
	2698,57112,94	C
	7140,80678,76154,20	$\frac{1}{4}$B
	7140,77980,19041,26	B
	37791,02218,15060,71453,80	$\frac{1}{2}$A
11	1,00000,37790,95077.37080,52412,54	A

places as there are ciphers at the beginning of the decimal in the number to be generated from the differences.

He then concludes this chapter with an ingenious, but not obvious, method of finding the differences B, C, D, E, &c, belonging to any number, as suppose the 9th, from that number itself, independent of any of the preceding 8th, 7th, 6th, 5th, &c; and it is this: raise the decimal A to the 2d, 3d, 4th, 5th, &c powers; then will the 2d (B), 3d (C), 4th (D), &c differences, be as here below, viz.

$B = \frac{1}{2}A^2$,

$C = \frac{1}{2}A^3 + \frac{1}{8}A^4$,

$D = \frac{7}{8}A^4 + \frac{7}{8}A^5 + \frac{7}{16}A^6 + \frac{1}{8}A^7 + \frac{1}{64}A^8$,

$E = \cdot\ 2\frac{5}{8}A^5 + 7A^6 + 10\frac{15}{16}A^7 + 12\frac{69}{128}A^8 + 11\frac{11}{64}A^9$ &c.

$F = \cdot\ \cdot\ 13\frac{9}{16}A^6 + 81\frac{3}{8}A^7 + 296\frac{87}{128}A^8 + 834\frac{43}{128}A^9$ &c.

$G = \cdot\ \cdot\ \cdot\ 122\frac{1}{16}A^7 + 1510\frac{67}{128}A^8 + 11475\frac{73}{128}A^9$ &c.

$H = \cdot\ \cdot\ \cdot\ \cdot\ 1937\frac{95}{128}A^8 + 47151\frac{93}{128}A^9$ &c.

$I = \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ 54902\frac{89}{128}A^9$ &c.

Thus in the 9th number of the foregoing example, omitting the ciphers at the beginning of the decimals, we have

$A =$ 1.51164,65999,05672,95048,8
$A^2 =$ - 2,28507,54430,06381,6726
$A^3 =$ - - 3,45422,65239,48546,2
$A^4 =$ - - - 5,22156,97802,288
$A^5 =$ - - - - 7,89316,8205
$A^6 =$ - - - - - 11,93168,1

Consequently,

$\frac{1}{2}A^2 =$	1.14253,77215,03190,8363	$= B$
$\frac{1}{2}A^3$	1,72711,32619,74273	
$\frac{1}{8}A^4$	- 65269,62225	
$\frac{1}{2}A^3 + \frac{1}{8}A^4$	1,72711,97889,36498	$= C$
$\frac{7}{8}A^4$	4,56887,35577	
$\frac{7}{8}A^5$	- 6,90652	
$\frac{7}{16}A^6$	- - 5	
$\frac{7}{8}A^4 + \frac{7}{8}A^5 + \frac{7}{16}A^6$	4,56894,26234	$= D$
$2\frac{5}{8}A^5$	- 20,71957	
$7A^6$	- - 83	
$2\frac{5}{8}A^5 + 7A^6$	- - 20,72040	$= E$

which agree with the like differences in the foregoing specimen.

In the 9th chapter, after observing that from the logarithms of 1, 2, 3, 5, and 10, before found, are to be determined, by addition and subtraction, the logarithms of all other numbers which can be produced from these by multiplication and division; for finding the logarithms of other prime numbers, instead of that in the 7th chapter, our author then shows another ingenious method of obtaining numbers beginning with 1 and ciphers, and such as to bear a certain relation to some prime number by means of which its logarithm may be found. The method is this: Find three products having the common difference 1, and such that two of them are produced from factors having given logarithms, and the third produced from the prime number, whose logarithm is required, either multiplied by itself, or by some other number whose logarithm is given: then the greatest and least of these three products being multiplied together, and the mean by itself, there arise two other products also differing by 1, of which the greater, divided by the less, gives for a quotient 1 with a small decimal, having several ciphers at the beginning. Then the logarithm of this quotient being found as before, from it will be deduced the required logarithm of the given prime number. Thus, if it be proposed to find the logarithm of the prime number 7; here $6 \times 8 = 48$, $7 \times 7 = 49$, and $5 \times 10 = 50$, will be the three products, of which the logarithms of 48 and 50, the 1st and 3d, will be given from those of their factors 6, 8, 5, 10: also $48 \times 50 = 2400$, and $49 \times 49 = 2401$ are the two new products, and $2401 \div 2400 = 1.00041\frac{2}{3}$ their quotient: then the least of 44 means between 1 and this quotient is 1.00000,00000,00000,02367,98249,04333,6405, which multiplied by 43429 &c, produces 0.00000,00000,00000,01028,40172,38387,29715 for its logarithm; which being 44 times doubled, or multiplied by 17592186044416, produces 0.00018,09183,45421,30 for the logarithm of the quotient $1.00041\frac{2}{3}$; which being added to the logarithm of the divisor 2400, gives the logarithm of the dividend 2401; then the half of this logarithm is the logarithm of 49 the root of 2401, and the half of this again gives 0.84509,80400,14256,82 for the logarithm of 7, which is the root of 49.—The author adds another example to illustrate this method; and then sets down the requisite factors, products, and quotients for finding the logarithms of all other prime numbers up to 100.

The 10th chapter is employed in teaching how to find the logarithms of fractions, namely by subtracting the logarithm of the denominator from that of the numerator, then the logarithm of the fraction is the remainder; which therefore is either abundant or defective, that is positive or negative, as the fraction is greater or less than 1.

In the 11th chapter is shown an ingenious contrivance for very accurately finding intermediate numbers to given logarithms, by the proportional parts. On this occasion, it is remarked, that while the absolute numbers increase uniformly, the logarithms increase unequally, with a decreasing increment; for which reason it happens, that either logarithms or numbers corrected by means of the proportional parts, will not be quite accurate, the logarithms so found being always too small, and the absolute numbers so found too great; but yet so however as that they approach much nearer to accuracy towards the end of the table, where the increments or differences become much nearer to equality, than in the former parts of the table. And from this property our author, ever fruitful in happy expedients to obviate natural difficulties, contrives a device to throw the proportional part, to be found from the numbers and logarithms, always near the end of the table, in whatever part they may happen naturally to fall. And it is this: Rejecting the characteristic of any given logarithm, whose number is proposed to be found, take the arithmetical complement of the decimal part, by subtracting it from 1.000&c, the logarithm of 10; then find in the table the logarithm next less than this arithmetical complement, together with its absolute number; to this tabular logarithm add the logarithm that was given, and the sum will be a logarithm necessarily falling among those near the end of the table; find then its absolute number, corrected by means of the proportional part, which will not be very inaccurate, as falling near the end of the table; this being divided by the absolute number, before found for the logarithm next less than the arithmetical complement, the quotient will be the required number answering to the given logarithm; which will be much more correct than if it had been found from the proportional part of the difference where it naturally happened to fall: and the reason of this operation is evident from the nature of logarithms. But as this divisor, when taken as the number answering to the logarithm next less than the arithmetical complement, may happen to be a large prime number; it is further remarked, that instead of this number and its logarithm, we may use the next less composite number, which has small factors, and *its* logarithms; because the division by those small factors, instead of by the number itself, will be performed by the short and easy way of division in one line. And for the more easy finding proper composite numbers and their factors, our author here subjoins an abacus, or list of all such numbers, with their logarithms and component factors, from 1000 to 10000; from which the proper logarithms and factors are immediately obtained by inspection. Thus, for example, to find the root of 10800, or the mean proportional between 1 and 10800: The logarithm of 10800 is 4.03342,37554,8695, the half of which is 2.01671,18777,4347 the logarithm of the number sought, the arithmetical complement of which log. is 0.98328,81222,5653; now the nearest log. to this in the abacus is 0.98227,12330,3957, and its annexed number is 9600, the factors of which are 2, 6, 8; to this last log. adding the log. of the number sought, the sum is 0.99898,31107,8304, whose absolute number, corrected by the proportional part, is 99766,12651,6521, which being divided continually by 2, 6, 8, the factors of 96, the last quotient is 103.92304845471; which is pretty correct, the true number being $103.923048454133 = \sqrt{10800}$.

We now arrive at the 12th and 13th chapters, in which our ingenious author first of all teaches the rules of the Differential Method, in constructing logarithms by interpolation from differences. This is the same method which has since been more largely treated of by later authors, and particularly by the learned Mr. Cotes, in his " Canonotechnia." How Mr. Briggs came by it does not well appear, as he only delivers the rules, without laying down the principles or investigation of them. He divides the method into two cases, namely, when the second differences are equal or nearly equal, and when the differences run out to any length whatever. The former of these is treated in the 12th chapter; and he particularly adapts it to the interpolating 9 equidistant means between two given terms, evidently for this reason, that then the powers of 10 become the principal multipliers or divisors, and so the operations performed mentally. The substance of his process is this: Having given two absolute numbers with their logarithms, to find the logarithms of 9 arithmetical means between the given numbers: Between the given logarithms take the 1st difference, as well as between each of them and their next or equidistant greater and less logarithms; and likewise the second differences, or the two differences of these three first differences; then if these second differences be equal, multiply one of them severally by the numbers 45, 35, &c, in the annexed tablet, dividing each product by 1000, that is cutting off three figures from each; lastly, to $\frac{1}{10}$ of the 1st difference of the given logarithms, add severally the first five quotients, and subtract the other five, so shall the ten results be the respective first differences, to be continually added, to compose the required series of logarithms.

1	45	Additive products.
2	35	
3	25	
4	15	
5	5	
6	5	Subductive products.
7	15	
8	25	
9	35	
10	45	

Now this amounts to the same thing as what is at this day taught in the like case: we know that if A be any term of an equidistant series of terms, and a, b, c, &c, the first of the 1st, 2d, 3d, &c, order of differences; then the term $\mathfrak{z}$, whose distance from A is expressed by x, will be thus, $z = A + xa + x \cdot \frac{x-1}{2} b + x \cdot \frac{x-1}{2} \cdot \frac{x-2}{3} c +$ &c. And if now, with our author, we make the 2d differences equal, then c, d, e, &c, will all vanish, or be equal to 0, and z will become barely $= A + xa + x \cdot \frac{x-1}{2} b$.

Series of Terms.	The Differences.
A	
$A + \frac{1}{10}a + \frac{9}{200}b$	$\frac{1}{10}a + \frac{9}{200}b = \frac{1}{10}a + \frac{45}{1000}b$
$A + \frac{2}{10}a + \frac{16}{200}b$	$\frac{1}{10}a + \frac{7}{200}b = \frac{1}{10}a + \frac{35}{1000}b$
$A + \frac{3}{10}a + \frac{21}{200}b$	$\frac{1}{10}a + \frac{5}{200}b = \frac{1}{10}a + \frac{25}{1000}b$
$A + \frac{4}{10}a + \frac{24}{200}b$	$\frac{1}{10}a + \frac{3}{200}b = \frac{1}{10}a + \frac{15}{1000}b$
$A + \frac{5}{10}a + \frac{25}{200}b$	$\frac{1}{10}a + \frac{1}{200}b = \frac{1}{10}a + \frac{5}{1000}b$
$A + \frac{6}{10}a + \frac{24}{200}b$	$\frac{1}{10}a - \frac{1}{200}b = \frac{1}{10}a - \frac{5}{1000}b$
$A + \frac{7}{10}a + \frac{21}{200}b$	$\frac{1}{10}a - \frac{3}{200}b = \frac{1}{10}a - \frac{15}{1000}b$
$A + \frac{8}{10}a + \frac{16}{200}b$	$\frac{1}{10}a - \frac{5}{200}b = \frac{1}{10}a - \frac{25}{1000}b$
$A + \frac{9}{10}a + \frac{9}{200}b$	$\frac{1}{10}a - \frac{7}{200}b = \frac{1}{10}a - \frac{35}{1000}b$
$A + a$	$\frac{1}{10}a - \frac{9}{200}b = \frac{1}{10}a - \frac{45}{1000}b$

Therefore if we take x successively equally to $\frac{0}{10}, \frac{1}{10}, \frac{2}{10}, \frac{3}{10}$, &c, we shall have the annexed series of terms with their differences. Where it is to be observed, that our author had reduced the differences from the 1st to the 2d form, as he thought it easier to multiply by 5 than to divide by 2. Also all the last terms $(x \cdot \frac{x-1}{2} b)$ are set down positive, because in the logarithms b is negative.—If the two 2d differences be only nearly equal, take an arithmetical mean between them, and proceed with it the same as above with one of the equal 2d differences.—He also shows how to find any one single term, independent of the rest; and concludes the chaper with pointing out a method of finding the proportional part more accurately than before.

In the 13th chapter our author remarks, that the best way of filling up the intermediate chiliads of his table, namely from 20000 to 90000, is by quinquisection, or interposing four equidistant means between two given terms; the method of performing this he thus particularly describes. Of the given terms, or logarithms, and two or three others on each side of them, take the 1st, 2d, 3d, &c, differences, till the last differences come out equal, which suppose to be the 5th differences: divide the first differences by 5, the 2d by 25, the 3d by 125, the 4th by 625, and the 5th by 3125, and call the respective quotients the 1st, 2d, 3d, 4th, 5th *mean* differences; or, instead of dividing by these powers of 5, multiply by their reciprocals $\frac{2}{10}$, $\frac{4}{100}$, $\frac{8}{1000}$, $\frac{16}{10000}$, $\frac{32}{100000}$; that is, multiply by 2, 4, 8, 16, 32, cutting off respectively one, two, three, four, five figures, from the end of the products, for the several mean differences: then the 4th and 5th of these mean differences are sufficiently accurate; but the 1st, 2d, and 3d are to be corrected in this manner; from the mean third differences subtract 3 times the 5th difference, and the remainders are the *correct* 3d differences; from the mean 2d differences subtract double the 4th differences, and the remainders are the correct 2d differences; lastly, from the mean 1st differences take the correct 3d differences, and $\frac{1}{5}$ of the 5th difference, and the remainders will be the correct first differences. Such are the corrections when the differences extend as far as the 5th. However, in completing those chiliads in this way, there will be only 3 orders of differences, as neither the 4th nor 5th will enter the calculation, but will vanish through their smallness: therefore the mean 2d and 3d differences will need no correction, and the mean first differences will be corrected by barely subtracting the 3d from them. These preparatory numbers being thus found, all the 2d differences of the logarithms required, will be generated by adding continually, from the less to the greater, the constant 3d difference; and the series of 1st differences will be found by adding the several 2d differences; and lastly, by adding continually these 1st differences to the 1st given logarithm &c, the required logarithmic terms will be generated.

These easy rules being laid down, Mr. Briggs next teaches how, by them, the remaining chiliads may best be completed: namely, having here the logarithm for all numbers up to 20000, find the logarithm to every 5 beyond this, or of 20005, 20010, 20015, &c, in this manner; to the logarithms of the 5th part of each of these, namely 4001, 4002, 4003, &c, add the constant logarithm of 5, and the sums will be the logarithms of all the terms of the series 20005, 20010, 20015, &c: and these logarithms will have the very same differences as those of the series 4001, 4002, 4003, &c; by means of which therefore interpose 4 equidistant terms by the rules above; and thus the whole canon will be easily completed.

Briggs here extends the rules for correcting the mean differences in quinquisection, as far as the 20th difference; he also lays down similar rules for trisection, and speaks of general rules for any other section, but omitted as being less easy. So that he appears to have been possessed of all that Cotes afterwards delivered in his "Canonotechnia sive Constructio Tabularum per Differentias," drawn from the Differential Method, as their general rules exactly agree, Briggs's mean and correct differences being by Cotes called round and quadrat differences, because he expresses them by the numbers 1, 2, 3, &c, written respectively within a small circle and square.

Briggs also observes, that the same rules equally apply to the construction of equidistant terms of any other kind, such as sines, tangents, secants, the powers of numbers, &c: and further remarks, that, of the sines of three equidifferent arcs, all the remote differences may be found by the rule of proportion, because the sines and their 2d, 4th, 6th, 8th, &c differences, are continued proportionals, as are also the 1st, 3d, 5th, 7th, &c differences, among themselves; and, like as the 2d, 4th, 6th, &c differences are proportional to the sines of the mean arcs, so also are the 1st, 3d, 5th, &c differences proportional to the cosines of the same arcs. Moreover, with regard to the powers of numbers, he remarks the following curious properties; 1st, that they will each have as many orders of differences as are denoted by the index of the power, the squares having two orders of differences, the cubes three, the 4th powers four, &c; 2d, that the last differences will be all equal, and each equal to the common difference of the sides or roots raised to the given power, and multiplied by $1\times2\times3\times4$ &c, continued to as many terms as there are units in the index: so, if the roots differ by 1, the second difference of the squares will be each 1×2 or 2, the 3d differences of the cubes each $1\times2\times3$ or 6, the 4th differences of the 4th powers each $1\times2\times3\times4$ or 24, and so on; and if the common difference of the roots be any other number n, then the last differences of the squares, cubes, 4th powers, 5th powers, &c, will be respectively $2n^2$, $6n^3$, $24n^4$, $120n^5$, &c.

Besides what was shown in the 11th chapter, concerning the taking out the logarithms of large numbers by means of proportional parts, Briggs employs the next or 14th chapter in teaching how, from the first ten chiliads only, and a small table of one page, here given, to find the number answering to any logarithm, and the logarithm to any number, consisting of fourteen places of figures*.

Having thus fully shown the construction and chief properties of his logarithms, our ingenious author, in the remaining eighteen chapters, exemplifies their uses in many curious and important subjects; such as The Rule-of-Three, or Rule of Proportion; finding the roots of given numbers; finding any number of mean proportionals between two given terms; with other arithmetical rules: also various geometrical subjects, as 1st, Having given the sides of any plane triangle, to find the area, the perpendicular, the angles, and the diameters of the inscribed and circumscribed circles; 2d, In a right-angled triangle, having given any two of these, to find the rest, viz. one leg and the hypotenuse, one leg and the sum or difference of the hypotenuse and the other leg, the two legs, one leg and the area, the area and the sum or difference of the legs, the hypotenuse and sum or difference of the legs, the hypotenuse and area, and the perimeter and area; 3d, Upon a given base, to describe a triangle, equal and isoperimetrical

* It is no more than a large exemplification of this method of Briggs's that has been printed so late as 1771, in a 4to tract, by Mr. Robert Flower, under the title of "The Radix, A New Way of making Logarithms." Though Briggs's work might not be known to this writer.

to another triangle given; 4th, To describe the circumference of a circle so, that the three distances from any point in it, to the three angles of a given plane triangle, shall be to one another in a given ratio; 5th, Having given the base, the area, and the ratio of the two sides, of a plane triangle, to find the sides; 6th, Given the base, difference of the sides, and area of a triangle, to find the sides; 7th, To find a triangle whose area and perimeter shall be expressed by the same number; 8th, Of four given lines, of which the sum of any three is greater than the fourth, to form a quadrilateral figure about which a circle may be described; 9th, Of the diameter, circumference, and area of a circle, and the surface and solidity of the sphere generated by it, having any one given, to find any one of the rest; 10th, Concerning the ellipse, spheroid, and gauging; 11th, To cut a line or a number in extreme and mean ratio; 12th, Given the diameter of a circle, to find the sides and areas of the inscribed and circumscribed regular figures of 3, 4, 5, 6, 8, 10, 12, and 16 sides; 13th, Concerning the regular figures of 7, 9, 15, 24, and 30 sides; 14th, Of isoperimetrical regular figures; 15th, Of equal regular figures; and 16th, Of the sphere and the 5 regular bodies; which closes this introduction. Such of these problems as can admit of it, are determined by elegant geometrical constructions, and they are all illustrated by accurate arithmetical calculations, performed by logarithms; for the exemplification of which they are purposely given.

At the end he remarks, that the chief and most necessary use of logarithms, is in the doctrine of spherical trigonometry, which he here promises to give in a future work, and which was accomplished in his Trigonometria Britannica, to the description of which we now proceed.

Of Briggs's Trigonometria Britannica.

At the close of the account of writings on the natural sines, tangents, and secants, we omitted the description of this work of our learned author, though it is perhaps the greatest of this kind, all things considered, that ever was executed by one person; purposely reserving the account of it to this place, not only as it is connected with the invention and construction of logarithms, but thinking it deserved more peculiar and distinguished notice, on account of the importance and originality of its contents. In the first place, we observe that the division of the quadrant, and the mode of construction, are both new; also the numbers are far more accurate, and are extended to more places, than they had ever been before. The circular arcs had always been divided in a sexagesimal proportion; but here the quadrant is divided into degrees and decimals, as this is a much easier mode of computation than by 60ths; the division being completed only to 100ths of degrees, though his design was to have extended it to 1000ths of degrees. And, besides his own private opinion, he was induced to adopt this mode of decimal divisions, partly at the request of other persons, and partly perhaps from the authority of Vieta, pa. 29 " Calendarii Gregoriani." And it is probable that computations by this decimal division would have come into general use, had it not been for the publication of Vlacq's tables, which came out in the interval, and were extended to every 10 seconds, or 6th parts of minutes. But besides this method, by a decimal division of the degrees, of which the whole circle contains 360, or the quadrant 90, in the 14th chapter he remarks that some other persons were inclined rather to adopt a complete decimal division of the whole circle, first into 100 parts, and each of these into 1000 parts; and for *their* sakes he subjoins a small table of the sines of every 40th part of the quadrant, and remarks, that from these few the whole may be made out, by continual quinquisections; namely, 5 times these 40 make 200, then 5 times these give 1000, thirdly 5 times these give 5000, and lastly, 5 times these give 25000 for the whole quadrant, or 100000 for the whole circumference.

But to return. Our author's large table consists of natural sines to 15 places, natural tangents and secants each to 10 places, logarithmic sines to 14 places, and logarithmic tangents to 10 places each, beside the characteristic. A most stupendous performance! The table is preceded by an introduction, divided into two books, the one containing an account of the truly ingenious construction of the table, by the author himself; and the other, its uses in trigonometry, &c, by Henry Gellibrand, professor of astronomy in Gresham College, who remarks in the preface, that the work was composed by the author about the year 1600; though it was only published by the direction of Gellibrand in 1633, it having been printed at Gouda under the care of Vlacq, and by the printer of his Trigonometria Artificialis, which came out the same year.

After briefly mentioning the common methods of dividing the quadrant, and constructing the tables of sines, &c, from the ancients down to his own time, he hastens to the description of his own peculiar and truly ingenious method, which is briefly this: having first divided the quadrant into a small number of parts, as 72, he finds the sine of one of those parts; then from it, the sines of the double, triple, quadruple, &c, up to the quadrant or 72 parts. He next quinquisects each of these parts, by interposing four equidistant means, by differences; he then quinquisects each of these; and finally each of these again; which completes the division as far as degrees and centesms. The rules for performing all these things he investigates, and illustrates, in a very ample manner. In treating of multiple and submultiple arcs, he gives general algebraical expressions for the sine or chord of any multiple whatever of a given arc, which he deduced from a geometrical figure, by finding the law for the series of successive multiple chords or sines, after the manner of Vieta; who was the first person that I know of, who laid down general rules for the chords of multiples and submultiples of arcs or angles: and the same was afterwards improved by Sir I. Newton, to such form, that radius, and double the cosine of the first given angle, are the first and second terms of all the proportions for finding the sines and cosines of the multiple angles. For assigning the coefficients of the terms in the multiple expressions, our author here delivers the construction of figurate or polygonal numbers, inserts a large table of them, and teaches their several uses; one of which is, that every other number, taken in the diagonal lines, furnishes the coefficients of the terms of the general equation, by which the sines and chords of multiple arcs are expressed, which he amply illustrates; and another, that the same diagonal numbers constitute the coefficients of the terms of any power of a binomial; which property was also mentioned by Vieta in his Angulares Sectiones, theor. 6, 7; and, before him, pretty fully treated of by Stifelius, in his Arithmetica Integra, fol. 44 et seq.; where he inserts and makes the like use of such a table of figurate numbers, in extracting the roots of all powers whatever. But it was perhaps known much earlier, as appears by the treatise on figurate numbers by Nicomachus, (see Malcolm's History, p. xviii). Though indeed Cardan seems to ascribe this discovery to Stifelius. See his Opus Novum de Proportionibus Numerorum, where he quotes it, and extracts the table and its use from Stifel's book. Cardan, in p. 135 &c, of the same work, makes use of a like table to find the number of variations, or conjugations, as he calls them. Stevinus too makes use of the same coefficients and method of roots as Stifelius. See his Arith. page 25. And even Lucas de Burgo extracts the cube root by the same coefficients, about the year 1470: but he does not go to any higher roots. And this is the first mention I have seen made of this law of the coefficients of the powers of a binomial, commonly called Sir I. Newton's binomial theorem, though it is very evident that Sir Isaac was not the first inventor of it: the part of it properly belonging to him seems to be, only the extending it to fractional indices, which was indeed an immediate effect of the general method of denoting all roots like powers, with fractional exponents, the theorem being not at all altered. However, it appears that our author Briggs was the first who taught the rule for generating the coefficients of the terms, successively one from another, of any power of a binomial, independent of those of any other power. For having shown, in his "Abacus Παγχρηστος" (which he so calls on account of its frequent and excellent use, and of which a small specimen is here annexed), that the numbers

ABACUS ΠΑΓΧΡΗΣΤΟΣ.							
H	G	F	E	D	C	B	A
−⑧	−⑦	+⑥	+⑤	−④	−③	+○	①
1	1	1	1	1	1	1	1
9	8	7	6	5	4	3	2
	36	28	21	15	10	6	3
		84	56	35	20	10	4
			126	70	35	15	5
				126	56	21	6
					84	28	7
						36	8
							9

in the diagonal directions, ascending from right to left, are the coefficients of the powers of binomials, the indices being the figures in the first perpendicular column A, which are also the coefficients of the 2d terms of each power (those of the first terms, being 1, are here omitted); and that any one of these diagonal numbers is in proportion to the next higher in the diagonal, as the vertical of the former is to the marginal of the latter, that is, as the uppermost number in the column of the former is to the first or right-hand number in the line of the latter; having shown these things, I say, he thereby teaches the generation of the coefficients of any power, independently of all other powers, by the very same law or rule which we now use in the binomial theorem. Thus, for the 9th power; 9 being the coefficient of the 2d term, and 1 always that of the first, to find the 3d coefficient, we have 2 : 8 : : 9 : 36; for the 4th term, 3 : 7 : : 36 : 84; for the 5th term, 4 : 6 : : 84 : 126; and so on for the rest. That is to say, the coefficients of the terms in any power m, are inversely as the vertical numbers or first line 1, 2, 3, 4, . . m, and directly as the ascending numbers m, $m-1$, $m-2$, $m-3$, . . . 1, in the first column A; and that consequently those coefficients are found by the continual multiplication of these fractions $\frac{m}{1}$, $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, ... $\frac{1}{m}$, which is the very theorem as it stands at this day, and as applied by Newton to roots or fractional exponents, as it had before been used for integral powers. This theorem then being thus plainly taught by Briggs about the year 1600, it is surprizing how a man of such general reading as Dr. Wallis was, could be quite ignorant of it, as he plainly appears to be by the 85th chapter of his algebra, where he fully ascribes the invention to Newton, and adds, that he himself had formerly sought for such a rule, but without success: Or how Mr. John Bernoulli, in the 18th century, could himself first dispute the invention of this theorem with Newton, and then give the discovery of it to Pascal, who was not born till long after it had been taught by Briggs. See Bernoulli's Works, vol. 4, page 173. But it is not to be wondered that Briggs's remark was unknown to Newton, who owed almost every thing to genius, and deep meditation, but very little to reading: and there can be no doubt that he made the discovery himself, without any light from Briggs; and that he thought it was new for all powers in general, as it was indeed for roots and quantities with fractional and irrational exponents.

When the above table of the sums of figurate numbers is used by our author, in determining the coefficients of the terms of the equation, whose root is the chord of any submultiple of an arc, as when the section is expressed by any uneven number, he remarks, that the powers of that chord or root will be the 1st, 3d, 5th, 7th, &c, in the alternate uneven columns, A, C, E, G, &c, with their signs + or − as marked to the powers, continued till the highest power be equal to the index of the section; and that the coefficients of those powers are the sums of two continuous numbers in the same column with the powers, beginning with 1 at the highest power, and gradually descending one line obliquely to the right at each lower power: so, for a trisection, the numbers are 1 in C, and 1 + 2 = 3 in A; and therefore the terms are − 1(3) + 3(1): for a quinquisection, the numbers are 1 in E, 1 + 4 = 5 in C, 2 + 3 = 5 in A; so that the terms are 1(5) − 5(3) + 5(1): for a septisection, the numbers are 1 in G, 1 + 6 = 7 in E, 4 + 10 = 14 in C, and 3 + 4 = 7 in A; and hence the terms are − 1 (7) + 7 (5) − 14 (3) + 7 (1): and so on; the sum of all these terms being always equal to the chord of the whole or multiple arc. But when the section is denominated by an even number, the squares of the chords enter the equation, instead of the first powers as before, and the dimensions of all the powers are doubled, the coefficients being found as before, and therefore the powers and numbers will be those in the 2d, 4th, 6th, &c, columns: and the uneven sections may also be expressed the same way: hence, for a bisection the terms will be − 1 (4) + 4 (2); for a trisection 1 (6) − 6 (4) + 9 (2); for the quadrisection − 1 (8) + 8 (6) − 20 (4) + 16 (2); for the quinquisection 1 (10) − 10 (8) + 35 (6) − 50 (4) + 25 (2); and so on.

Our author subjoins another table, a small specimen of which is here annexed, in which the first column consists of the uneven numbers 1, 3, 5, &c, the rest being found by addition as before, and the alternate diagonal numbers themselves are the coefficients.

F	E	D	C	B	A
+ (6)	+ (5)	− (4)	− (3)	+ (2)	(1)
1	1	1	1	1	1
	7	6	5	4	3
		20	14	9	5
			30	16	7
				25	9
					11

The method is quite different from that of Vieta, who gives another table for the like purpose, a small part of which is here annexed, which is formed by adding, from the number 2, downwards obliquely towards the right; and the coefficients of the terms stand upon the horizontal line.

1st	Vieta's Table.				
2					
3	2d				
4	2				
5	5	3d			
6	9	2			
7	14	7	4th		
8	20	16	2		
9	27	30	9	5th	
10	35	50	25	2	6th

These angular sections were afterwards further discussed by Oughtred and Wallis. And the same theorems of Vieta and Briggs have been since given in a different form, by Herman and the Bernoullis, in the Leipsic Acts, and the Memoirs of the Royal Academy of Sciences. These theorems they expressed by the alternate terms of the power of a binomial, whose exponent is that of the multiple angle or section. And De Lagny, in the same Memoirs, first showed, that the tangents and secants of multiple angles are also expressed by the terms of a binomial, in the form of a fraction, of which some of those terms form the numerator, and others the denominator. Thus, if r express the radius, s the sine, c the cosine, t the tangent, and s the secant, of the angle A; then the sine, cosine, tangent, and secant of n times the angle, are expressed thus, viz.

$$\text{Sin. } n\text{A} = \frac{1}{r^{n-1}} \times \left(\frac{n}{1}c^{n-1}s - \frac{n.n-1.n-2}{1.2.3}c^{n-3}s^3 + \frac{n.n-1.n-2.n-3.n-4}{1.2.3.4.5}c^{n-5}s^5 \,\&c\right)$$

$$\text{Cosine } n\text{A} = \frac{1}{r^{n-1}} \times \left(c^n - \frac{n.n-1}{1.2}c^{n-2}s^2 + \frac{n.n-1.n-2.n-3}{1.2.3.4}c^{n-4}s^4 \,\&c.\right)$$

$$\text{Tang. } n\text{A} = r \times \frac{\frac{n}{1}r^{n-1}t - \frac{n.n-1.n-2}{1.2.3}r^{n-3}t^3 + \frac{n.n-1.n-2.n-3.n-4}{1.2.3.4.5}r^{n-5}t^5 \,\&c.}{r^n - \frac{n.n-1}{1.2}r^{n-2}t^2 + \frac{n.n-1.n-2.n-3}{1.2.3.4}r^{n-4}t^4 \,\&c.}$$

$$\text{Sec. } n\text{A} = r \times \frac{\text{s}^2 \text{ or } r^2 + t^2}{r^n - \frac{n.n-1}{1.2}r^{n-2}t^2 + \frac{n.n-1.n-2.n-3}{1.2.3.4}r^{n-4}t^4 \,\&c.} :$$

where it is evident, that the series in the sine of n A, consists of the even terms of the power of the binomial $(c + s)^n$, and the series in the cosine of the uneven terms of the same power; also the series in the numerator of the tangent, consists of the even terms of the power $(r + t)^n$, and the denominator, both of the tangent and secant, consists of the uneven terms of the same power $(r + t)^n$. And if the diameter, chord, and chord of the supplement, be substituted for the radius, sine and cosine, in the expressions for the multiple sine and cosine, the result will give the chord, and chord of the supplement, of n times the arc or angle A. These, and various other expressions, for multiple and submultiple arcs, with other improvements in trigonometry, have also been given by Euler, and other eminent writers on the same subject.

The before mentioned De Lagny offered a project for substituting, instead of the common logarithms, a binary arithmetic, which he called the *natural logarithms*, and which he and Leibnitz seem to have both invented about the same time, independently of each other: but the project came to nothing. De Lagny also published, in several Memoirs of the Royal Academy, a new method of determining the angles of figures, which he called *Goniometry*. It consists in measuring, with a pair of compasses, the arc which subtends the angle in question: however, this arc is not measured in the usual way, by applying its extent to any preconstructed scale; but by examining what part it is of half the circumference of the same circle, in this manner: from the proposed angular point as a centre, with a sufficiently large radius, a semicircle being described, a part of which is the arc intercepted by the sides of the proposed angle, the extent of this arc is taken with a pair of fine compasses, and applied continually upon the arc of the semicircle, by which he finds how often it is contained in the semicircle, with usually a small arc remaining; in the same manner he measures how often this remaining arc is contained in the first arc; and what remains again is applied continually to the first remainder; and so the 3d remainder to the 2d, the 4th to the 3d, and so on till there be no remainder, or else till it become insensibly small. By this process he obtains a series of quotients, or fractional parts, one of another, which being properly reduced into one fraction, give the ratio of the first arc to the semicircumference, or of the proposed angle, to two right angles or 180 degrees, and consequently that angle in degrees, minutes, &c, if required, and that commonly, he says, to a degree of accuracy far exceeding the calculation of the same by means of any tables of sines, tangents or secants, notwithstanding the apparent paradox in this expression at first sight. Thus, if the 1st arc be 4 times contained in the semicircle, the remainder once contained in the first arc, the next 5 times in the second, and finally the fourth 2 times in the third: Here the quotients are 4, 1, 5, 2; consequently the fourth or last arc was $\frac{1}{2}$ the 3d; therefore the 3d was $\frac{1}{5\frac{1}{2}}$ or $\frac{2}{11}$ of the 2d, and the 2d was $\frac{1}{1\frac{2}{11}}$ or $\frac{11}{13}$ of the 1st, and the first, or arc sought, was $\frac{1}{4\frac{11}{13}}$ or $\frac{13}{63}$ of the semicircle; and consequently it contains $37\frac{1}{7}$ degrees, or $37° 8' 34''\frac{2}{7}$. Hence it is evident, that this method is in fact nothing more than an example of continued fractions, the first instance of which was given by lord Brouncker.

But to return from this long digression; Mr. Briggs next treats of interpolation by differences, and chiefly of quinquisection, after the manner used in the 13th chapter of his construction of logarithms, before described. He here proves that curious property of the sines and their several orders of differences, before mentioned, namely, that, of equidifferent arcs, the sines, with the 2d, 4th, 6th, &c differences, are continued proportionals; as also the cosines of the means between those arcs, and the 1st, 3d, 5th, &c differences. And to this treatise on interpolation by differences, he adds a marginal note, complaining that this 13th chapter of his "Arithmetica Logarithmica" had been omitted by Vlacq in his edition of it; as if he were afraid of an intention to deprive him of the honour of the invention of interpolation by successive differences. The note is this: "Modus correctionis à me traditus est Arithmeticæ Logarithmicæ capite 13, in editione Londinensis: Istud autem caput unà cum sequenti in editione Batava me inconsulto et inscio omissum fuit: nec in omnibus, editionis illius author, vir alioqui industrius et non indoctus, meam mentem videtur assequutus: Ideoque, ne quicquam desit cuiquam, qui integrum canonem conficere cupiat, quædam maxime necessaria illinc huc transferenda censui."

A large specimen of quinquisection by differences is then given, and he shows how it is to be applied to the construction of the whole canon of sines, both for 100th and 1000th parts of degrees; namely, for centesms, divide the quadrant first into 72 equal parts, and find their sines by the primary methods; then these quinquisected give 360 parts, a second quinquisection gives 1800 parts, and a third gives 9000 parts, or centesms of degrees: but for millesms, divide the quadrant into 144 equal parts; then one quinquisection gives 720, a second gives 3600, a third 18000, and a fourth gives 90000 parts, or millesms.

He next proceeds to the natural tangents and secants, which he directs to be raised in the same manner, by interpolations from a few primary ones, constructed from the known proportions between sines, tangents, and secants; excepting that half the tangents and secants are to be formed by addition and subtraction only, by means of some such theorems as these, namely, 1st, the secant of an arc is equal to the sum of the tangent of the same arc, and the tangent of half its complement, which will find every other secant; 2d, double the tangent of an arc added to the tangent of half its complement, is equal to the tangent of the sum of that arc and the said half complement, by which rule half the tangents will be found; &c.

In the two remaining chapters of this book are treated the construction of the logarithmic sines, tangents, and secants. This is preceded by some remarks on the origin and invention of them. Our author here observes, that logarithms may be of various kinds; that others had followed the plan of Baron Napier the first inventor, among whom Benjamin Ursinus is especially commended, who applied Napier's logarithms to every ten seconds of the quadrant; but that he himself, encouraged by the noble inventor, devised other logarithms that were much easier and more excellent*. He says he put 10, with ciphers, for the logarithm of radius; 9 for the logarithm sine of 5° 44′, whose natural sine is one 10th of the radius; 8 for that of 34′, whose natural sine is one 100th of the radius, and so on; thereby making 1 the loga-

* His words are: " Ego vero ipsius inventoris primi cohortatione adjutus, alios logarithmos applicandos censui, qui multo faciliorem usum habent, præstantiorem. Logarithmus radii circularis vel sinus totius, a me ponitur 10 &c."

rithm of the ratio of 10 to 1, which is the characteristic of his species of logarithms.

To construct the logarithmic sines, he directs first to divide the quadrant into 72 equal parts as before, and to find the logarithms of their natural sines as in the 14th chapter of his Arithmetica Logarithmica; after which, this number will be increased by quinquisection, first to 360, then to 1800, and lastly to 9000, or centesms of degrees. But if millesms of degrees be required, divide the quadrant first into 144 equal parts, and then by four quinquisections these will be extended to the following parts, 720, 3600, 18000, and 90000, or millesms of degrees. He remarks however, that the logarithmic sines of only half the quadrant need be found in this manner, as the other half may be found by mere addition, or subtraction, by means of this theorem, as the sine of half an arc is to half radius, so is the sine of the whole arc to the cosine of the said half arc. This theorem he illustrates with examples, and then adds a table of the logarithmic sines of the primary 72 parts of the quadrant, from which the rest are to be made out by quinquisection.

In the next chapter our author shows the construction of the natural tangents and secants more fully than he had done before, demonstrating and illustrating several curious theorems for the easy finding of them. He then concludes this chapter, and the book, with pointing out the very easy construction of the logarithmic tangents and secants by means of these three theorems:

1st, As cosine : sine : : radius : tangent,
2d, As tangent : radius : : radius : cotangent,
3d, As cosine : radius : : radius : secant.

So that in logarithms, the tangents are found by subtracting the cosines from the sines, adding always 10 or the radius; the cotangents are found by subtracting always the tangents from 20 or double the radius; and the secants are found by subtracting the cosines from 20 the double radius.—The 2d book, by Gellibrand, contains the use of the canon in plane and spherical trigonometry.

Besides Briggs's methods of constructing logarithms, above described, no others were given about that time. For as to the calculations made by Vlacq, his numbers being carried to comparatively but few places of figures, they were performed by the easiest of Briggs's methods, and in the manner which this ingenious man had pointed out in his two volumes. Thus, the 70 chiliads of logarithms, from 20000 to 90000, computed by Vlacq, and published in 1628, being extended only to 10 places, yield no more than two orders of mean differences, which are also the correct differences, in quinquisection, and therefore will be made out thus, namely, one-fifth of them by the mere addition of the constant logarithm of 5; and the other four-fifths of them by two easy additions of very small numbers, namely, of the 1st and 2d differences, according to the directions given in Briggs's Arith. Log. c. 13, p. 31. And as to Vlacq's logarithmic sines and tangents to every 10 seconds, they were easily computed thus; the sines for half the quadrant were found by taking the logarithms to the natural sines in Rheticus's canon; and then from these the logarithmic sines to the other half quadrant were found by mere addition and subtraction; and from these all the tangents by one single subtraction. So that all these operations might easily be performed by one person, as quickly as a printer could set up the types; and thus the computation and printing might both be carried on together. And hence it appears that there is no reason for admiration at the expedition with which these tables were said to have been brought out.

Of certain curves related to Logarithms.

About this time the mathematicians of Europe began to consider some curves which have properties analogous to logarithms. Edmund Gunter, it has been said, first gave the idea of a curve, whose abscisses are in arithmetical progression, while the corresponding ordinates are in geometrical progression, or whose abscisses are the logarithms of their ordinates; but I cannot find it noticed in any part of his writings. The same curve was afterwards considered by others, and named the *Logarithmic* or *Logistic* curve by Huygens, in his " Dissertatio de Causa Gravitatis," where he enumerates all the principal properties of this curve, showing its analogy to logarithms. Many other learned men have also treated of its properties ; particularly Le Seur and Jacquier, in their commentary on Newton's Principia ; by Dr. John Keill, in the elegant little tract on logarithms, subjoined to his edition of Euclid's Elements ; and by Francis Maseres, Esq. cursitor baron of the exchequer, in his ingenious treatise on Trigonometry; in which books the doctrine of logarithms is copiously and learnedly treated, and their analogy to the logarithmic curve &c fully displayed.—It is indeed rather extraordinary that this curve was not sooner announced to the public; since it results immediately from baron Napier's manner of conceiving the generation of logarithms, by only supposing the lines which represent the natural numbers to be placed at right angles to that upon which the logarithms are taken. This curve greatly facilitates the conception of logarithms to the imagination, and affords an almost intuitive proof of the very important property of their fluxions, or very small increments, to wit, that the fluxion of the number is to the fluxion of the logarithm, as the number is to the subtangent; as also of this property, that, if three numbers be taken very nearly equal, so that their ratios to each other may differ but a little from a ratio of equality, as for exam. the three numbers 10000000, 10000001, 10000002, their differences will be very nearly proportional to the logarithms of the ratios of those numbers to each other: all which follows from the logarithmic arcs being very little different from their chords, when they are taken very small. And the constant subtangent of this curve is what was afterwards by Cotes called the *Modulus* of the system of logarithms: and since, by the former of the two properties above-mentioned, this subtangent is a 4th proportional to the fluxion of the number, the fluxion of the logarithm, and the number itself; this property afforded occasion to Mr. Baron Maseres to give the following definition of the modulus, which is the same in effect as Cotes's, but more clearly expressed, namely, that it is the limit of the magnitude of a 4th proportional to these three quantities, to wit, the difference of any two natural numbers that are nearly equal to each other, either of the said numbers, and the logarithm or measure of the ratio they have to each other. Or we may define the modulus to be the natural number at that part of the system of logarithms, where the fluxion of the number is equal to the fluxion of the logarithm, or where the numbers and logarithms have equal differences. And hence it follows, that the logarithms of equal numbers, or of equal ratios, in different systems, are to one another as the *moduli* of those systems. Further, the ratio whose measure or logarithm is equal to the modulus, and thence by Cotes called the *ratio modularis*, is by calculation found to be the ratio of 2·718281828459 &c to 1, or of 1 to ·367879441171 &c; the calculation of which number may be seen at full length in Mr. Baron Maseres's treatise on the Principles of Life Annuities, pa. 274 and 275.

The hyperbolic curve also afforded another source for developing and illustrating the properties and construction of logarithms. For the hyperbolic areas lying between the curve and one asymptote, when they are bounded by ordinates parallel to the other asymptote, are analogous to the logarithms of their abscisses, or parts of the asymptote. And so also are the hyperbolic sectors; any sector bounded by an arc of the hyperbola and two radii, being equal to the quadrilateral space bounded by the same arc, the two ordinates to either asymptote from the extremities of the arc, and the part of the asymptote intercepted between them. And though Napier's logarithms are commonly said to be the same as hyperbolic logarithms, it is not to be understood that hyperbolas exhibit Napier's logarithms only, but indeed all other possible systems of logarithms whatever. For, like as the right-angled hyperbola, the side of whose square inscribed at the vertex is 1, gives Napier's logarithms; so any other system of logarithms is expressed by the hyperbola whose asymptotes form a certain oblique angle, the side of the rhombus inscribed at the vertex of the hyperbola in this case also being still 1, the same as the side of the square in the right-angled hyperbola. But the areas of the square and rhombus, and consequently the logarithms of any one and the same number or ratio, differing according to the sine of the angle of the asymptotes. And the area of the square or rhombus, or any inscribed parallelogram, is also the same thing as what was by Cotes called the modulus of the system of logarithms; which modulus will therefore be expressed by the numerical measure of the sine of the angle formed by the asymptotes, to the radius 1; as that is the same with the number expressing the area of the said square or rhombus, the side being 1: which is another definition of the modulus to be added to those we remarked above, in treating of the logarithmic curve. And the evident reason of this is, that in the beginning of the generation of these areas, from the vertex of the hyperbola, the nascent increment of the abscisse drawn into the altitude 1, is to the increment of the area, as radius is to the sine of the angle of the ordinate and abscisse, or of the asymptotes; and at the beginning of the logarithms, the nascent increment of the natural numbers is to the increment of the logarithms, as 1 is to the modulus of the system. Hence we easily discover that the angle formed by the asymptotes of the hyperbola exhibiting Briggs's system of logarithms, will be 25 deg. 44 minutes, 25½ seconds, this being the angle whose sine is 0·4342944819 &c, the modulus of this system.

Or indeed any one hyperbola will express all possible systems of logarithms whatever, namely, if the square or rhombus inscribed at the vertex, or, which is the same thing, any parallelogram inscribed between the asymptotes and the curve at any other point, be expounded by the modulus of the system; or, which is the same, by expounding the area, intercepted between two ordinates which are to each other in the ratio of 10 to 1, by the logarithm of that ratio in the proposed system.

As to the first remarks on the analogy between logarithms and the hyperbolic spaces; it having been shown by Gregory

St. Vincent, in his Quadratura Circuli et Sectionum Coni, published at Antwerp in 1647, that if one asymptote be divided into parts, in geometrical progression, and from the points of division ordinates be drawn parallel to the other asymptote, they will divide the space between the asymptote and curve into equal portions; from hence it was shown by Mersenne, that, by taking the continual sums of those parts, there would be obtained areas in arithmetical progression, adapted to abscisses in geometrical progression, and which therefore were analogous to a system of logarithms. And the same analogy was remarked and illustrated soon after, by Huygens and many others, who showed how to square the hyperbolic spaces by means of the logarithms.

There are also innumerable other geometrical figures having properties analogous to logarithms; such as the equiangular spiral, the figures of the tangents and secants, &c; which it is not to our purpose to distinguish more particularly.

Of Gregory's Computation of Logarithms.

On the other hand, Mr. James Gregory, in his Vera Circuli et Hyperbolæ Quadratura, first printed at Patavi, or Padua, in the year 1667, having approximated to the hyperbolic asymptotic spaces by means of a series of inscribed and circumscribed polygons, thence shows how to compute the logarithms, which are analogous to those areas: and thus the quadrature of the hyperbolic spaces became the same thing as the computation of the logarithms. He here also lays down various methods to abridge the computation, with the assistance of some properties of numbers themselves, by which we are enabled to compose the logarithms of all prime numbers under 1000, each by one multiplication, two divisions, and the extraction of the square root. And the same subject is further pursued in his Exercitationes Geometricæ, to be described hereafter.

Mr. Gregory was born at Aberdeen in Scotland 1638, where he was educated. He was professor of mathematics in the college of St. Andrews, and afterwards in that of Edinburgh.

He died of a fever in December 1675, being only 36 years of age.

Of Mercator's Logarithmotechnia.

Nicholas Mercator, a learned mathematician, and an ingenious member of the Royal Society, was a native of Holstein in Germany, but spent most of his time in England, where he died in the year 1690, at about 50 years of age. He was the author of many works in geometry, geography, astronomy, astrology, &c.

In 1668, Mercator published his Logarithmotechnia, sive methodus construendi Logarithmos nova, accurata, et facilis; in which he delivers a new and ingenious method of computing the logarithms, on principles purely arithmetical; which, being curious and very accurately performed, I shall here give a rather full and particular account of that little tract, as well as of the small specimen of the quadrature of curves by infinite series, subjoined to it; and the more especially as this work gave occasion to the public communication of some of Sir Isaac Newton's earliest pieces, to evince that he had not borrowed them from this publication. So that it appears these two ingenious men had, independent of each other, in some instances fallen upon the same things.

Mercator begins this work by remarking that the word logarithm is composed of the words ratio and number, being as much as to say the number of ratios; which he observes is quite agreeable to the nature of them, for that a logarithm is nothing else but the number of *ratiunculæ* contained in the ratio which any number bears to unity. He then makes a learned and critical dissertation on the nature of ratios, their magnitude and measure, conveying a clearer idea of the nature of logarithms than had been given by either Napier or Briggs, or any other writer except Kepler, in his work before described; though those other writers seem indeed to have had in their own minds the same ideas on the subject as Kepler and Mercator, but without having expressed them so clearly. Our author indeed pretty closely follows Kepler in his modes of thinking and expression, and after him in plain and express terms calls logarithms the measures of ratios; and, in order to the right understanding that definition of them, he explains what he means by the magnitude of a ratio. This he does pretty fully, but not too fully, considering the nicety and subtlety of the subject of ratios, and their magnitude, with their addition to, and subtraction from, each other, which have been misconceived by very learned mathematicians, who have thence been led into considerable mistakes. Witness the oversight of Gregory St. Vincent, which Huygens animadverted on in the Εξετασις Cyclometriæ Gregorii a Sancto Vincentio, and which arose from not understanding, or not adverting to, the nature of ratios, and their proportions to one another. And many other similar mistakes might here be adduced of other eminent writers. From all which we must commend the propriety of our author's attention, in so judiciously discriminating between the magnitude of a ratio, as of a to b, and the fraction $\frac{a}{b}$, or quotient arising from the division of one term of the ratio by the other; which latter method of consideration is always attended with danger of errors and confusion on the subject; though in the 5th definition of the 6th book of Euclid this quotient is accounted the quantity of the ratio; but this definition is probably not genuine, and therefore very properly omitted by professor Simson in his edition of the Elements. And in those ideas on the subject of logarithms, Kepler and Mercator have been followed by Halley, Cotes, and most of the other eminent writers since that time.

Purely from the above idea of logarithms, namely, as being the measures of ratios, and as expressing the number of *ratiunculæ* contained in any ratio, or into which it may be divided, the number of the like equal *ratiunculæ* contained in some one ratio, as of 10 to 1, being supposed given, our author shows how the logarithm or measure of any other ratio may be found. But this however only by-the-bye, as not being the principal method he intends to teach, as his last and best, and which we arrive not at till near the end of the book, as we shall see below. Having shown then, that these logarithms, or numbers of small ratios, or measures of ratios, may be all properly represented by numbers, and that of 1, or the ratio of equality, the logarithm or measure being always 0, the logarithm of 10, or the measure of the ratio 10 to 1, is most conveniently represented by 1 with any number of ciphers; he then proceeds to show how the measures of all other ratios may be found from this last supposition. And he explains the principles by the two following examples.

First, to find the logarithm of 100·5*, or to find how many *ratiunculæ* are contained in the ratio of 100·5 to 1, the number of *ratiunculæ* in the decuple ratio, or ratio of 10 to 1, being 1.0000000.

The given ratio 100·5 to 1, he first divides into its parts, namely, 100·5 to 100, 100 to 10, and 10 to 1; the last two of which being decuples, it follows that the characteristic will be 2, and it only remains to find how many parts of the next decuple belong to the first ratio of 100·5 to 100. Now if each term of this ratio be multiplied by itself, the products will be in the duplicate ratio of the first terms, or this last ratio will contain a double number of parts; and if these be multiplied by the first terms again, the ratio of the last products will contain three times the number of parts; and so on, the number of times of the first parts contained in the ratio of any like powers of the first terms, being always denoted by the exponent of the power. If therefore the first terms, 100·5 and 100, be continually multiplied till the same powers of them have to each other a ratio whose measure is known, as suppose the decuple ratio 10 to 1, whose measure is 1.0000000; then the exponent of that power shows what mul. this measure 1.0000000, of the decuple ratio, is of the required measure of the first ratio 100·5 to 100; and consequently dividing 1.0000000 by that exponent, the quotient is the measure of the ratio 100·5 to 100 sought. The operation for finding this, he sets down as here follows; where the several multiplications are all performed in

* Mercator distinguishes his decimals from integers thus 100[5, or 100|5.

the contracted way, by inverting the figures of the multiple and retaining only the first number of decimals in each product.

	power
100·5000	. . 1
5001	. . 1
1005000	
5025	
1010025	. . 2
5200101	. . 2
1010025	
10100	
20	
5	
1020150	. . 4
0510201	. . 4
1020150	
20403	
102	
51	
1040706	. . 8
6070401	. . 8
1083068	. . 16
8603801	. . 16
1173035	. . 32
5303711	. . 32
1376011	. . 64
1106731	. . 64
1893406	. 128
6043981	. 128
3584985	. 256
5894853	. 256
12852116	. 512

This power being greater than the decuple of the like power of 100, which must always be 1 with ciphers, resume therefore the 256th power, and multiply it, not by itself, but by the next before, viz. by the 128th, thus

3584985	. 256
6043981	. 128
6787831	. 384
1106731	. 64
9340130	. 448
5303711	. 32
10956299	. 480

This power again exceeding the same power of 100 more than 10 times, therefore draw the same 448th, not into the 32d, but the next preceding, thus

9340130	. 448
8603801	. 16
10115994	. 464

This being again too much, instead of the 16th, draw it into the 8th, or next preceding, thus

9340130	. 448
6070401	. 8
9720329	. 456
0520201	. 4
9916193	. 460
5200101	. 2
10015603	. 462

Which power again exceeds the limit; theref. draw the 460th into the 1st, thus

9916193	. 460
5001	. 1
9965774	. 461

Since therefore the 462d power of 100·5 is greater, and the 461st power is less, than the decuple of the same power of 100, the ratio of 100·5 to 100 is contained in the decuple more than 461 times, but less than 462 times. Again,

Since the	power is		and the differences	
460		9916193		
461		9965774	49581	nearly
462		10015603	49829	equal;

therefore the proportional part which the exact power, or 10000000, exceeds the next less 9965774, will be easily and accurately found by the Golden Rule, thus:

The just power	. .	10000000
and the next less	. .	9965774
the difference	.	34226; then

As 49829 the dif. between the next less and greater,
: To 34226 the dif. between the next less and just,
: : So is 10000 : to 6868, the decimal parts; and therefore the ratio of 100·5 to 100, is 461·6868 times contained in the decuple or ratio of 10 to 1. Dividing now 1.0000000, the measure of the decuple ratio, by 461·6868, the quotient 00216597 is the measure of the ratio of 100·5 to 100; which being added to 2 the measure of 100 to 1, the sum 2.00216597 is the measure of the ratio of 100·5 to 1, that is, the log. of 100·5 is 2.00216597.—In the same manner he next investigates the log. of 99·5, and finds it to be 1.99782307.

A few observations are then added, calculated to generalize the consideration of ratios, their magnitude and affections. It is here remarked, that he considers the magnitude of the ratio between two quantities as the same, whether the antecedent be the greater or the less of the two terms: so, the magnitude of the ratio of 8 to 5, is the same as of 5 to 8; that is, by the magnitude of the ratio of either to the other, is meant the number of *ratiunculæ* between them, which will evidently be the same, whether the greater or less term be the antecedent. And he further remarks that, of different ratios, when we divide the greater term of each ratio by the less, that ratio is of the greater mass or magnitude, which produces the greater quotient, *et vice versa;* though those quotients are not proportional to the masses or magnitudes of the ratios. But when he considers the ratio of a greater term to a less, or of a less to a greater, that is to say, the ratio of greater or less inequality, as abstracted from the magnitude of the ratio, he distinguishes it by the word *affection,* as much as to say, greater or less affection, something in the manner of positive and negative quantities, or such as are affected with the signs

+ and −. The remainder of this work he delivers in several propositions, as follows.

Prop. 1. In subtracting from each other, two quantities of the same affection, to wit, both positive, or both negative; if the remainder be of the same affection with the two given, then is the quantity subtracted the less of the two, or expressed by the less number; but if the contrary, it is the greater.

Prop. 2. In any continued ratios, as $\frac{a}{a+b}$, $\frac{a+b}{a+2b}$, $\frac{a+2b}{a+3b}$, &c, (by which is meant the ratios of a to $a+b$, $a+b$ to $a+2b$, $a+2b$ to $a+3b$, &c,) of equidifferent terms, the antecedent of each ratio being equal to the consequent of the next preceding one, and proceeding from less terms to greater; the measure of each ratio will be expressed by a greater quantity than that of the next following; and the same through all their orders of differences, namely, the 1st, 2d, 3d, &c, differences; but the contrary, when the terms of the ratios decrease from greater to less.

Prop. 3. In any continued ratios of equidifferent terms, if the 1st or least be a, the difference between the 1st and 2d b, and c, d, e, &c, the respective first term of their 2d, 3d, 4th, &c, differences: then shall the several quantities themselves be as in the annexed scheme; where each term is composed of the first term, together with as many of the differences as it is distant from the first term, and to those differences joining, for coefficients, the numbers in the sloping or oblique lines contained in the annexed table of figurate numbers, in the same

1st term a
2d term $a + b$
3d term $a + 2b + c$
4th term $a + 3b + 3c + d$
5th term $a + 4b + 6c + 4d + e$
&c. &c.

1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9
1	3	6	10	15	21	28	36	
1	4	10	20	35	56	84		
1	5	15	35	70	126			
1	6	21	56	126				
1	7	28	84					
1	8	36						
1	9							

1st term	a
2d term	$a - b$
3d term	$a - 2b + c$
4th term	$a - 3b + 3c - d$
5th term	$a - 4b + 6c - 4d + e$
&c.	&c.

manner, he observes, as the same figurate numbers complete the powers raised from a binomial root, as had long before been taught by others. He also remarks, that this rule not only gives any one term, but also the sum of any number of successive terms from the beginning, making the 2d coefficient the first, the 3d the 2d, and so on; thus, the sum of the first 5 terms is $5a + 10b + 10c + 5d + e$.

In the 4th *prop.* it is shown, that if the terms decrease, proceeding from the greater to the less, the same theorems hold good, by only changing the sign of every other term, as in the margin.

Prop. 5 shows how to find any multiple nearly of a given ratio. To do this, take the difference of the terms of the ratio, which multiply by the index of the multiple, from the product subtract the same difference; add half the remainder to the greater term of the ratio, and subtract the same half from the less term, which give two terms expressing the required multiple a little less than the truth.—Thus, to quadruple the ratio $\frac{25}{28}$: the difference of the terms 3 multiplied by 4 makes 12, from which 3 deducted leaves 9, its half $4\frac{1}{2}$ added to the greater term 28 makes $32\frac{1}{2}$, and taken from the less term 25, leave $20\frac{1}{2}$; then $20\frac{1}{2}$ and $32\frac{1}{2}$ are the terms nearly of the quadruple sought, or reduced to whole numbers gives $\frac{41}{65}$, a little less than the truth.

Prop. 6 and 7 treat of the approximate multiplication and division of ratios, or, which is the same thing, the finding nearly any powers or any roots of a given fraction, in an easy manner. The theorem for raising any power, when reduced to a simpler form, is this, the m power of $\frac{a}{b}$, or $(\frac{a}{b})^m$, is $= \frac{s \mp md}{s \pm md}$ nearly, where s is $= a + b$, and $d = a \backsim b$, the sum and difference of the two numbers, and the upper or under signs take place according as $\frac{a}{b}$ is a proper or an improper fraction, that is, according as a is less or greater than b. And the th. for extracting the mth root of $\frac{a}{b}$, is $\sqrt[m]{\frac{a}{b}}$ or $(\frac{a}{b})^{\frac{1}{m}} = \frac{ms \mp d}{ms \pm d}$ nearly; which latter rule is also the same as the former, as will be evident by substituting $\frac{1}{m}$ instead of m in the first theorem. So that universally $(\frac{a}{b})^{\frac{m}{n}}$, is $= \frac{ns \mp md}{ns \pm md}$ nearly. These theorems however are nearly true only in some certain cases, namely when $\frac{a}{b}$ and $\frac{m}{n}$ do not differ greatly from unity. And in the 7th *prop.* the author shows how to find nearly the error of the theorems.

In the 8th *prop.* it is shown, that the measures of ratios of equidifferent terms, are nearly reciprocally as the arithmetical means between the terms of each ratio. So, of the ratios $\frac{16}{18}$, $\frac{33}{35}$, $\frac{50}{52}$, the mean between the terms of the first ratio is 17, of the 2d 34, of the 3d 51, and the measure of the ratios are nearly as $\frac{1}{17}$, $\frac{1}{34}$, $\frac{1}{51}$.

From this property he proceeds, in the 9th *prop.* to find the measure of any ratio less than $\frac{99 \cdot 5}{100 \cdot 5}$, which has an equal difference, 1, of terms. In the two examples, mentioned near the beginning, our author found the logarithm, or measure of the ratio, of $\frac{99 \cdot 5}{100}$, to be $21769\frac{3}{10}$, and that of $\frac{100}{100 \cdot 5}$ to be $21659\frac{7}{10}$; therefore the sum 43429 is the logarithm of $\frac{99 \cdot 5}{100 \cdot 5}$, or $\frac{99 \cdot 5}{100} \times \frac{100}{100 \cdot 5}$; or the logarithm of $\frac{99 \cdot 5}{100 \cdot 5}$ is nearer 43430, as found by other more accurate computations. Now to find the logarithm of $\frac{100}{101}$, having the same difference of terms, 1, with the former; it will be, by prop. 8, as 100·5 (the mean between 101 and 100) : 100 (the mean between 99·5 and 100·5) : : 43430 : 43213 the logarithm of $\frac{100}{101}$, or the difference between the logarithms of 100 and 101. But the log. of 100 is 2; therefore the logarithm of 101 is 2.0043213.—Again, to find the logarithm of 102, we must first find the logarithm of $\frac{101}{102}$; the mean between its terms being 101·5, therefore as 101·5 : 100 : : 43430 : 42788 the logarithm of $\frac{101}{102}$, or the difference of the logarithms of 101 and 102. But the logarithm of 101 was found above to be 2.0043213; therefore the log. of 102 is 2.0086001.—So that, dividing continually 868596 (the double of 434298 the logarithm of $\frac{99.5}{100.5}$ or $\frac{199}{201}$) by each number of the series 201, 203, 205, 207, &c, then add 2 to the 1st quotient, to the sum add the 2d quotient, and so on, adding always the next quotient to the last sum, the several sums will be the respective logarithms of the numbers in this series 101, 102, 103, 104, &c.

The next, or *prop.* 10, shows that, of two pair of continued ratios, whose terms have equal differences, the difference of the measures of the first two ratios, is to the difference of the measures of the other two, as the square of the common term in the two latter, is to that in the former, nearly. Thus, in the four ratios $\frac{a}{a+b}$, $\frac{a+b}{a+2b}$, $\frac{a+3b}{a+4b}$, $\frac{a+4b}{a+5b}$; as the measure of $\frac{aa+2ab}{(a+b)^2}$ (the difference of the first two, or the quotient of the two fractions) : is to the measure of $\frac{aa+8ab+15bb}{(a+4b)^2}$: : so $(a+4b)^2$: is to $(a+b)^2$, nearly.

In *prop.* 11 the author shows that similar properties take place among two sets of ratios consisting each of 3 or 4 &c continued numbers.

Prop. 12 shows that, of the powers of numbers in arithmetical progression, the orders of differences which become equal, are the 2d differences in the squares, the 3d differences in the cubes, the 4th differences in the 4th powers, &c. And hence it is shown, how to construct all those powers by the continual addition of their differences; as had been long before more fully explained by Briggs.

In the next, or 13th *prop.* our author explains his compendious method of raising the tables of logarithms; showing how to construct the logarithms by addition only, from the properties contained in the 8th, 9th, and 12th props. For this purpose, he makes use of the quantity $\frac{a}{b-c}$, which by division he resolves into this infinite series $\frac{a}{b}+\frac{ac}{bb}+\frac{ac^2}{b^3}+\frac{ac^3}{b^4}$ &c (*in infin.*). Putting then $a=100$, the arithmetical mean between the terms of the ratio $\frac{99\cdot5}{100\cdot5}$, $b = 100000$, and c successively equal to 0·5, 1·5, 2·5, &c, that so $b - c$ may be respectively equal to 99999·5, 99998·5, 99997·5, &c, the corresponding means between the terms of the ratios $\frac{99999}{100000}$, $\frac{99998}{99999}$, $\frac{99997}{99998}$, &c, it is evident that $\frac{a}{b-c}$ will be the quotient of the 2d term divided by the 1st, in the proportions mentioned in the 8th and 9th propositions; and when all of these quotients are found, it remains then only to multiply them by the constant 3d term 43429, or rather 43429·8, of the proportion, to produce the logarithms of the ratios $\frac{99999}{100000}$, $\frac{99998}{99999}$, $\frac{99997}{99998}$, &c, till $\frac{10000}{10001}$; then adding these continually to 4, the logarithm of 10000, the least number, or subtracting them from 5, the logarithm of the highest term 100000, there will result the logarithms of all the absolute numbers from 10000 to 100000. Now when $c = 0\cdot5$, then

$$\frac{a}{b} = \cdot001,\ \frac{ac}{bb} = \cdot000000005,\ \frac{ac^2}{b^3} = \cdot000000000000025,\ \frac{ac^3}{b^4} = \cdot000000000000000000125$$

&c; therefore $\frac{a}{b-c} = \frac{a}{b} + \frac{ac}{bb} + \frac{ac^2}{b^3}$ &c, is $= \cdot00100000050000025000125$,

In like manner, if $c = 1\cdot5$, then $\frac{a}{b-c}$ will be $= \cdot001000001500002250003375$,

and if $c = 2\cdot5$, then $\frac{a}{b-c}$ will be $= \cdot00100000250000625015625$;

&c. But instead of constructing all the values of $\frac{a}{b-c}$ in the usual way of raising the powers, he directs them to be found by addition only, as in the last proposition. Having thus found all the values of $\frac{a}{b-c}$, the author then shows, that they may be drawn into the constant logarithm 43429 by addition only, by the help of the annexed table of its first 9 products.

1	43429
2	86858
3	130287
4	173716
5	217145
6	260574
7	304003
8	347432
9	390861

The author then distinguishes which of the logarithms it may be proper to find in this way, and which from their component parts. Of these, the logarithms of all even numbers need not be thus computed, being composed from the number 2; which cuts off one-half of the numbers: neither are those numbers to be computed which end in 5, because 5 is one of their factors;

these last are $\frac{1}{10}$ of the numbers; and the two together $\frac{1}{2}$ + $\frac{1}{10}$ make $\frac{3}{5}$ of the whole: and of the other $\frac{2}{5}$, the $\frac{1}{3}$ of them, or $\frac{2}{15}$ of the whole, are composed of 3; and hence $\frac{3}{5} + \frac{2}{15}$, or $\frac{11}{15}$ of the numbers, are made up of such as are composed of 2, 3, and 5. As to the other numbers which may be composed of 7, of 11, &c; he recommends to find *their* logarithms in the general way, the same as if they were incomposites, as it is not worth while to separate them in so easy a mode of calculation. So that of the 90 chiliads of numbers, from 10000 to 100000, only 24 chiliads are to be computed.—Neither indeed are all of these to be calculated from the foregoing series for $\frac{a}{b-c}$, but only a few of them in that way, and the rest by the proportion in the 8th proposition. Thus, having computed the logarithms of 10003 and 10013, omitting 10023, as being divisible by 3, estimate the logarithms of 10033 and 10043, which are the 30th numbers from 10003 and 10013; and again omitting 10053, a multiple of 3, find the logarithms of 10063 and 10073. Then by prop. 8,

As 10048, the arithmetical mean between 10033 and 10063,
to 10018, the arithmetical mean between 10003 and 10033,
so 13006, the dif. between the logs. of 10003 and 10033,
to 12967, the dif. between the logs. of 10033 and 10063.

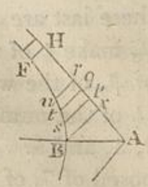
$$\text{That is, 1st, As} \begin{cases} 10048 \\ 10078 \\ 10108 \end{cases} : 10018 :: 13006 : \begin{cases} 12967 \\ \&c. \\ \end{cases}$$

$$\text{Again, As} \begin{cases} 10058 \\ 10088 \\ 10118 \end{cases} : 10028 :: 12992 : \begin{cases} 12953 \\ \&c. \\ \end{cases}$$

$$\text{And 3dly, As} \begin{cases} 10068 \\ 10098 \\ \&c. \end{cases} : 10038 :: 12979 : \begin{cases} 12940 \\ \&c. \\ \end{cases}$$

And with this our author concludes his compendium for constructing the tables of logarithms.

He afterwards shows some applications and relations of the doctrine of logarithms to geometrical figures: in order to which, in *prop.* 14, he proves algebraically that, in the right-angled hyperbola, if from the vertex, and from any other



point, there be drawn BI, FH perpendicular to the asymptote AH, or parallel to the other asymptote; then will AH : AI : : BI : FH. And,

In *prop.* 15, if AI = BI = 1, and HI = a; then will FH $= \frac{1}{1+a} = 1 - a + a^2 - a^3 + a^4 - a^5$ &c, *in infinitum*, by a continued algebraic division, the process of which he describes, step by step, as a thing that was new or uncommon. But that method of division had been taught before, by Dr. Wallis in his Opus Arithmeticum.

Prop. 16 is this: Any given number being supposed to be divided into innumerable small equal parts, it is required to assign the sum of any powers of the continual sums of those innumerable parts. For which purpose he lays down this rule; if the next higher power of the given number, above that power whose sum is sought, be divided by its exponent, the quotient will be the sum of the powers sought. That is, if N be the given number, and a one of its innumerable equal parts, then will

$a^n + (2a)^n + (3a)^n + (4a)^n$ &c . . . . N^n be $= \frac{N^{n+1}}{n+1}$: which theorem he demonstrates by a method of induction. And this, it is evident, is the finding the sum of any powers of an infinite number of arithmeticals, of which the greatest term is a given quantity, and the least indefinitely small. It is also remarkable, that the above expression is similar to the rule for finding the fluent to the given fluxion of a power, as afterwards taught by Sir I. Newton.

Mercator then applies this rule, in *prop.* 17, to the quadrature of the hyperbola. Thus, putting AI = 1, conceive the asymptote to be divided from I into innumerable equal parts, namely I$p = pq = qr = a$; then, by the 14th and 15th

$$\left.\begin{array}{l} ps = 1 - a + a^2 - a^3 \text{ \&c} \\ qt = 1 - 2a + 4a^2 - 8a^3 \text{ \&c} \\ ru = 1 - 3a + 9a^2 - 27a^3 \text{ \&c} \end{array}\right\}$$

But the area BIru is = the sum $ps + qt + ru$, which is =

$3 - 6a + 14a^2 - 36a^3$ &c, that is, equal to the number of terms contained in the line Ir, minus the sum of those terms, plus the sum of the squares of the same, minus the sum of their cubes, plus the sum of the 4th powers, &c. Putting now IA = 1, as before, and I*p* = 0·1 the number of terms, to find the area BI*ps*; by prop. 16 the sum of the terms will be $\frac{0 \cdot 1^2}{2}$ = ·005, the sum of their squares = ·000333333, the sum of their cubes ·000025, the sum of the 4th powers ·000002, the sum of the 5th powers ·000000166, the sum of the 6th powers ·000000014, &c. Therefore the area BI*ps* is = ·1 − ·005 + ·000333333 − ·000025 + ·000002 − ·000000166 + ·000000014 &c = ·100335347 − ·005025166 = ·095310181 &c.

Again, putting I*q* = ·21 the number of terms, he finds in like manner the area BI*qt* = ·21 − ·02205 + ·003087 − ·000486202 + ·000081682 − ·000014294 + ·000002572 − ·000000472 + ·000000088 &c = ·213171345 − ·022550984 = ·190620361 &c.

He then adds, hence it appears that, as the ratio of AI to A*p*, or 1 to 1·1, is half or subduplicate of the ratio of AI to A*q*, or 1 to 1·21, so the area BI*ps* is here found to be half of the area BI*qt*. These areas he computes to 44 places of figures, and finds them still in the ratio of 2 to 1.

The foregoing doctrine amounts to this, that if the rectangle BI × I*r*, which in this case is expressed by I*r* only, be put = A, AI being = 1, as before; then the area BI*ru*, or the hyperbolic logarithm of 1 + A, or of the ratio of 1 to 1 + A, will be equal to the infinite series $A - \frac{1}{2}A^2 + \frac{1}{3}A^3 - \frac{1}{4}A^4 + \frac{1}{5}A^5$ &c; and which therefore may be considered as Mercator's quadrature of the hyperbola, or his general expression of an hyperbolic logarithm in an infinite series. And this method was further improved by Dr. Wallis in the Philos. Trans. for the year 1668.

In *prop.* 18 Mercator compares the hyperbolic *areolæ* with the *ratiunculæ* of equidifferent numbers, and observes that, the areola BI*ps* is the measure of the ratiuncula of AI to A*p*, the areola *spqt* is the measure of the ratiuncula of A*p* to A*q*, the areola *tqru* is the measure of the ratiun. of A*q* to A*r*, &c.

Finally, in the 19th *prop.* he shows how the sums of logarithms may be taken, after the manner of the sums of the *areolæ*. And hence infers, as a corollary, how the continual product of any given numbers in arithmetical progression may be obtained: for the sum of the logarithms is the logarithm of the continual product. He then remarks, that from the premises it appears, in what manner Mersennus's problem may be resolved, if not geometrically, at least in figures to any number of places. And thus closes this ingenious tract.

In the Philos. Trans. for 1668 are also given some further illustrations of this work, by the author himself. And in various places also in a similar manner are logarithms and hyperbolic areas treated of by Lord Brouncker, Dr. Wallis, Sir I. Newton, and many other learned persons.

Of Gregory's Exercitationes Geometricæ.

In the same year 1668 came out Mr. James Gregory's Exercitationes Geometricæ, in which are contained the following pieces:

1, Appendicula ad veram circuli et hyperbolæ quadraturam:

2, N. Mercatoris quadratura hyperbolæ geometricè demonstrata:

3, Analogia inter lineam meridianam planisphærii nautici et tangentes artificiales geometricè demonstrata; seu quod secantium naturalium additio efficiat tangentes artificiales:

4, Item, quot tangentium naturalium additio efficiat secantes artificiales:

5, Quadratura conchoidis:

6, Quadratura cissoidis: et

7, Methodus facilis et accurata componendi secantes et tangentes artificiales.

The first of these pieces, or the Appendicula, contains some further extension and illustration of his Vera circuli et hyperbolæ quadratura, occasioned by the animadversions made on that work by the celebrated mathematician and philosopher Huygens.

In the 2d is demonstrated geometrically, the quadrature of the hyperbola; by which he finds a series similar to Mercator's for the logarithm, or the hyperbolic space beyond the first ordinate (BI, fig. pa. 416). In like manner he finds another series for the space at an equal distance within that ordinate. These two series having all their terms alike, but all the signs of the one plus, and those of the other alternately plus and minus, by adding the two together, every other term is cancelled, and the double of the rest denotes the sum of both spaces. Gregory then applies these properties to the logarithms; the conclusion from all which may be thus briefly expressed:

$$\text{since } A - \tfrac{1}{2}A^2 + \tfrac{1}{3}A^3 - \tfrac{1}{4}A^4 \text{ \&c} = \text{the log. of } \frac{1+A}{1},$$

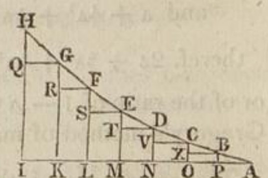
$$\text{and } A + \tfrac{1}{2}A^2 + \tfrac{1}{3}A^3 + \tfrac{1}{4}A^4 \text{ \&c} = \text{the log. of } \frac{1}{1-A},$$

$$\text{theref. } 2A + \tfrac{2}{3}A^3 + \tfrac{2}{5}A^5 + \tfrac{2}{7}A^7 \text{ \&c} = \text{the log. of } \frac{1+A}{1-A},$$

or of the ratio of $1 - A$ to $1 + A$. Which may be accounted Gregory's method of making logarithms.

The remainder of this little volume is chiefly employed about the nautical meridian, and the logarithmic tangents and secants. It does not appear by whom, nor by what accident, was discovered the analogy between a scale of logarithmic tangents and Wright's protraction of the nautical meridian line, which consisted of the sums of the secants. It appears however to have been first published, and introduced into the practice of navigation, by Henry Bond, who mentions this property in an edition of Norwood's Epitome of Navigation, printed about 1645; and he again treats of it more fully in an edition of Gunter's works, printed in 1653, where he teaches, from this property, to resolve all the cases of Mercator's sailing by the logarithmic tangents, independent of the table of meridional parts. This analogy had only been found to be nearly true by trials, but not demonstrated to be a mathematical property. Such demonstration seems to have been first discovered by Nicholas Mercator, who, desirous of making the most advantage of this and another concealed invention of his in navigation, by a paper in the Philos. Trans.

for June 4, 1666, invites the public to enter into a wager with him, on his ability to prove the truth or falsehood of the supposed analogy. This mercenary proposal however seems not to have been taken up by any one, and Mercator reserved his demonstration. The proposal however excited the attention of mathematicians to the subject itself, and a demonstration was not long wanting. The first was published about two years after by Gregory, in the tract now under consideration, and from thence and other similar properties, here demonstrated, he shows, in the last article, how the tables of logarithmic tangents and secants may easily be computed, from the natural tangents and secants. The substance of which is as follows:



Let AI be the arc of a quadrant, extended in a right line, and let the figure AHI be composed of the natural tangents of every arc from the point A, erected perpendicular to AI at their respective points: let AP, PO, ON, NM, &c, be the very small equal parts into which the quadrant is divided, namely, each $\frac{1}{60}$, or $\frac{1}{100}$ of a degree; draw PB, OC, ND, ME, &c, perpendicular to AI. Then it is manifest, from what had been demonstrated, that the figures ABP, ACO, &c, are the artificial secants of the arcs AP, AO, &c, putting O for the artificial radius. It is also manifest, that the rectangles BO, CN, DM, &c, will be found from the multiplication of the small part AP of the quadrant by each natural tangent. But, he proceeds, there is a little more difficulty in measuring the figures ABP, BCX, CDV, &c; for if the first differences of the tangents be equal, AB, BC, CD, &c, will not differ from right lines, and then the figures ABP, BCX, CDV, &c, will be right-angled triangles, and therefore any one, as HQG, will be $=$ $\frac{1}{2}$QH × QG: but if the second differences be equal, the said figures will be portions of trilineal quadratices; for example, HQG will be a portion of a trilineal quadratix, whose axis is parallel to QH; and each of the last differences being z, it will be $QHG = \frac{1}{2}QH \times QG - \frac{1}{12}Z \times QG$; and if the 3d differences be equal, the said figures will be portions of trilineal cubices, and then shall QHG be equal $\frac{1}{2}QH \times QG - \sqrt{(\frac{1}{72}QH \times Z \times QG^2 - \frac{1}{1728}Z^2 \times QG^2)}$: when the 4th differences are equal, the said figures are portions of trilineal quadrato-quadratices, and the 4th differences are equal to 24 times the 4th power of QG, divided by the cube of the latus rectum; also when the 5th differences are equal, the said figures are portions of trilineal sursolids, and the 5th differences are equal to 120 times the sursolid of QG, divided by the 4th power of the latus rectum; and so on *in infinitum*. What has been here said of the composition of artificial secants from the natural tangents, it is remarked, may in like manner be understood of the composition of artificial tangents, from the natural secants, according to what was before demonstrated. It is also observed, that the artificial tangents and secants are computed, as above, on the supposition that 0 is the log. of 1, and 1000000000000 the radius, and 23025850929940456240178 70 the log. of 10; but that they may be more easily computed, namely by addition only, by putting $\frac{1}{60}$ of a degree $= QG = AP = 1$, and the logarithm of $10 = 7915704467897819$; for by this means $\frac{1}{2}QH \times QG$ is $= \frac{1}{2}QH = QHG$, and $\frac{1}{2}QH \times QG - \frac{1}{12}Z \times QG = \frac{1}{2}QH - \frac{1}{12}Z = QHG$, also $\frac{1}{2}QH \times QG - \sqrt{(\frac{1}{72}QH \times Z \times QG^2 - \frac{1}{1728}Z^2 \times QG^2)}$ $= \frac{1}{2}QH - \sqrt{(\frac{1}{72}QH \times Z - \frac{1}{1728}Z^2)} = QHG$: And finally, by one division only are found the artificial tangents and secants to 1000000000000000, the logarithm of 10, putting still 1 for radius, which are the differences of the artificial tangents and secants, in the table, from that artificial radius; and to make the operations easier in multiplying by the number 7915704467897819, or logarithm of 10, a table is set down of its products by the first 9 figures. But if AP or QG be $= \frac{1}{100}$ of a degree, the artificial tangents and secants will answer to 13192840779829703 as the logarithm of 10, the first 9 multiples of which are also placed in the table. But to represent the numbers by the artificial radius, rather than by the logarithm of 10, the author directs to add ciphers, &c.—And so much for Gregory's Exercitationes Geometricæ.

The same analogy between the logarithmic tangents and the meridian line, as also other similar properties, were afterwards more elegantly demonstrated by Dr. Halley in the Philos. Trans. for Feb. 1696, and various methods given for computing the same, by examining the nature of the spirals into which the rhumbs are transformed in the stereographical projection of the sphere, on the plane of the equator: the doctrine of which was rendered still more easy and elegant by the ingenious Mr. Cotes, in his Logometria, first printed in the Philos. Trans. for 1714, and afterwards in the collection of his works published in 1722, by his cousin Dr. Robert Smith, who succeeded him in the Plumian professorship of philosophy in the University of Cambridge.

The learned Dr. Isaac Barrow also, in his Lectiones Geometricæ, lect. xi. Append. first published in 1672, delivers a similar property, namely, that the sum of all the secants of any arc is analogous to the logarithm of the ratio of $r + s$ to $r - s$, or radius plus sine to radius minus sine; or, which is the same thing, that the meridional parts answering to any degree of latitude, are as the logarithms of the ratios of the versed sines of the distances from the two poles.

Mr. Gregory's method for making logarithms was further exemplified in numbers, in a small tract on this subject, printed in 1688, by one Euclid Speidell, a simple and illiterate person, and son of John Speidell, before mentioned among the first writers on logarithms.

Gregory also invented many other infinite series, and among them these here following, viz. a being an arc, t its tangent, and s the secant, to the radius r; then is

$$a = t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} \text{ \&c.}$$

$$t = a + \frac{a^3}{3r^2} + \frac{2a^5}{15r^4} + \frac{17a^7}{315r^6} + \frac{62a^9}{2835r^8} \text{ \&c.}$$

$$s = r + \frac{a^2}{2r} + \frac{5a^4}{24r^3} + \frac{61a^6}{720r^5} + \frac{277a^8}{8064r^7} \text{ \&c.}$$

And if τ and σ denote the artificial or logarithmic tangent and secant of the same arc a, the whole quadrant being q, and $e = 2a - q$; then is

$$e = \tau - \frac{\tau^3}{6r^2} + \frac{\tau^5}{24r^4} - \frac{61\tau^7}{5040r^6} + \frac{277\tau^9}{72576r^8} \text{ \&c.}$$

$$\tau = e + \frac{e^3}{6r^2} + \frac{e^5}{24r^4} + \frac{61e^7}{5040r^6} + \frac{277e^9}{72576r^8} \text{ \&c.}$$

$$\sigma = \frac{a^2}{2r} + \frac{a^4}{12r^3} + \frac{a^6}{45r^5} + \frac{17a^8}{2520r^7} + \frac{62a^{10}}{28350r^9} \text{ \&c.}$$

And if s denote the artificial secant of 45°, and s + l the artificial secant of any arc a, the artificial radius being 0; then is

$$a = \tfrac{1}{2}q + l - \frac{l^2}{r} + \frac{4l^3}{3r^2} - \frac{7l^4}{3r^3} + \frac{14l^5}{3r^4} - \frac{452l^6}{45r^5} \text{ \&c.}$$

The investigation of all which series may be seen at pa. 298 et seq. vol. 1, Dr. Horsley's commentary on Sir I. Newton's works, as they were given in the Commercium Epistolicum, no. xx, without demonstration, and where the number 2 is also wanting in the denominator of the first term of the series expressing the value of σ.

Such then were the ways in which Mercator and Gregory applied these their very simple series $A - \frac{1}{2}A^2 + \frac{1}{3}A^3 - \frac{1}{4}A^4$ &c, and $A + \frac{1}{2}A^2 + \frac{1}{3}A^3 + \frac{1}{4}A^4$ &c, for the purpose of computing logarithms. But they might, as I apprehend, have applied them to this purpose in a shorter and more direct manner, by computing, by their means, only a few logarithms of small ratios, in which the terms of the series would have decreased by the powers of 10, or some greater number, the numerators of all the terms being unity, and their denominators the powers of 10 or some greater number, and then employing these few logarithms, so computed, to the finding the logarithms of other and greater ratios, by the easy operations of mere addition and subtraction. This might have been done for the logarithms of the ratios of the first ten numbers, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11, to 1, in the following manner, communicated by Mr. Baron Maseres.

In the first place, the logarithm of the ratio of 10 to 9, or of 1 to $\frac{9}{10}$, or of 1 to $1 - \frac{1}{10}$, is equal to the series

$$\frac{1}{10} + \frac{1}{2 \times 100} + \frac{1}{3 \times 1000} + \frac{1}{4 \times 10000} + \frac{1}{5 \times 100000} \text{ \&c.}$$

In like manner are easily found the logarithms of the ratios of 11 to 10; and then, by the same series, those of 121 to 120, and of 81 to 80, and of 2401 to 2400; in all which cases the series would converge still faster than in the two first cases. We may then proceed by mere addition and subtraction of logarithms, as follows;

Log. $\frac{11}{9}$ = L. $\frac{11}{10}$ + L. $\frac{10}{9}$,	L. $\frac{10}{4}$ = L. $\frac{10}{9}$ + L. $\frac{9}{4}$,
L. $\frac{121}{81}$ = 2L. $\frac{11}{9}$,	L. $\frac{81}{16}$ = 2L. $\frac{9}{4}$,
L. $\frac{121}{80}$ = L. $\frac{121}{81}$ + L. $\frac{81}{80}$,	L. $\frac{80}{16}$ = L. $\frac{81}{16}$ − L. $\frac{81}{80}$,
L. $\frac{120}{80}$ = L. $\frac{121}{80}$ − L. $\frac{121}{120}$,	L. $\frac{5}{1}$ = L. $\frac{80}{16}$,
L. $\frac{120}{80}$ = L. $\frac{3}{2}$,	L. $\frac{5}{2}$ = L. $\frac{10}{4}$,
L. $\frac{9}{4}$ = 2L. $\frac{3}{2}$,	L. $\frac{2}{1}$ = L. $\frac{5}{1}$ − L. $\frac{5}{2}$.

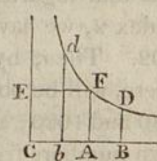
Having thus got the logarithm of the ratio of 2 to 1, or, in common language, the logarithm of 2, the logarithms of all sorts of even numbers may be derived from those of the odd numbers, which are their coefficients, with 2 or its powers. We may then proceed as follows:

L. 4 = 2L. 2,	L. 24 = L. 8 + L. 3,
L. 10 = L. $\frac{10}{4}$ + L. 4,	L. 2400 = L. 100 + L. 24,
L. 9 = L. $\frac{9}{4}$ + L. 4,	L. 2401 = L. $\frac{2401}{2400}$ + L. 2400,
L. 3 = $\frac{1}{2}$L. 9,	L. 7 = $\frac{1}{4}$L. 2401,
L. 100 = 2L. 10,	L. 11 = L. $\frac{11}{9}$ + L. 9,
L. 8 = 3L. 2,	L. 6 = L. 2 + L. 3.

Thus we have got the logarithms of 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11. And this is, upon the whole, perhaps the best method of computing logarithms that can be taken. There have been indeed some methods discovered by Dr. Halley, and other mathematicians, for computing the logarithms of the ratios of prime numbers, to the next adjacent even numbers, which are still shorter than the application of the foregoing series. But those methods are less simple and easy to understand, and apply, than these series; and the computation of logarithms by these series, when their terms decrease by the powers of 10, or of some greater number, is so very short and easy (as we have seen in the foregoing computations of the logarithms of the ratios of 10 to 9, 11 to 10, 81 to 80, 121 to 120, &c,) that it is not worth while to seek for any shorter methods of computing them. And this method of computing logarithms is very nearly the same with that of Sir I. Newton, in his second letter to Mr. Oldenburg, dated October 1676, as will be seen in the following article.

Of Sir Isaac Newton's Methods.

The excellent Sir I. Newton greatly improved the quadrature of the hyperbolical-asymptotic spaces by infinite series, derived from the general quadrature of curves by his method of fluxions; or rather indeed he invented that method himself, and the construction of logarithms derived from it, in the year 1665 or 1666, before the publication of either Mercator's or Gregory's books, as appears by his letter to Mr. Oldenburg dated October 24, 1676, printed in p. 634 *et seq.* vol. 3, of Wallis's works, and elsewhere. The quadrature of the hyperbola, thence translated, is to this effect. Let dFD be an hyperbola, whose centre is C, vertex F, and interposed square CAFE $=$ 1. In CA take AB and Ab on each side $= \frac{1}{10}$ or 0·1: And, erecting the perpendiculars BD, bd; half the sum of the spaces AD and Ad will be $= 0{\cdot}1 + \frac{0{\cdot}001}{3} + \frac{0{\cdot}00001}{5} + \frac{0{\cdot}0000001}{7}$ &c. and the half diff. $= \frac{0{\cdot}01}{2} + \frac{0{\cdot}0001}{4} + \frac{0{\cdot}000001}{6} + \frac{0{\cdot}00000001}{8}$ &c.



Which reduced will stand thus,

1·0000000000000	,0·0050000000000	The sum of these 0·1053605156577 is Ad,
3333333333	250000000	and the differ. 0·0953101798043 is AD.
20000000	1666666	In like manner, putting AB and Ab
142857	12500	each = 0·2, there is obtained
1111	100	Ad = 0·2231435513142, and
9	1	AD = 0·1823215567939.
0·1003353477310	,0·0050251679267	

Having thus the hyperbolic logarithms of the four decimal numbers 0·8, 0·9, 1·1, and 1·2; and since $\frac{1{\cdot}2}{0{\cdot}8} \times \frac{1{\cdot}2}{0{\cdot}9} = 2$, and 0·8 and 0·9 are less than unity; adding their logarithms to double the logarithm of 1·2, we have 0·6931471805597, the hyperbolic logarithm of 2. To the triple of this adding the log. of 0·8, because $\frac{2 \times 2 \times 2}{0{\cdot}8} = 10$, we have 2·3025850929933, the logarithm of 10. Hence by one addition are found the logarithms of 9 and 11: And thus the logarithms of all these prime numbers, 2, 3, 5, 11 are prepared. Further, by only depressing the numbers, above computed, lower in the decimal places, and adding, are obtained the logarithms of the decimals 0·98, 0·99, 1·01, 1·02; as also of these 0·998, 0·999, 1·001, 1·002. And hence, by addition and subtraction, will arise the logarithms of the primes 7, 13, 17, 37, &c. All which logarithms being divided by the above logarithm of 10, give the common logarithms to be inserted in the table.

And again, a few pages further on, in the same letter, he resumes the construction of logarithms, thus: Having found, as above, the hyperbolic logarithms of 10, 0·98, 0·99, 1·01, 1·02, which may be effected in an hour or two, dividing the last four logarithms by the logarithm of 10, and adding the index 2, we have the tabular logarithms of 98, 99, 100, 101, 102. Then, by interpolating nine means between each of these, will be obtained the logarithms of all numbers between 980 and 1020; and again interpolating 9 means between every two numbers from 980 to 1000, the table will be so far constructed. Then from these will be collected the logarithms of all the primes under 100, together with those of their multiples: all which will require only addition and subtraction; for

$$\sqrt[10]{\frac{9984 \times 1020}{9945}}=2;\ \frac{10}{2}=5;\ \sqrt{\frac{98}{2}}=7;\ \frac{99}{9}=11;\ \frac{1001}{7 \times 11}=13;\ \frac{102}{6}=17;$$

$$\frac{988}{4 \times 13}=19;\ \frac{9936}{16 \times 27}=23;\ \frac{986}{2 \times 17}=29;\ \frac{992}{32}=31;\ \frac{999}{27}=37;\ \frac{984}{24}=41;$$

$$\frac{989}{23}=43;\ \frac{987}{27}=47;\ \frac{9911}{11 \times 17}=53;\ \frac{9971}{13 \times 13}=59;\ \frac{9882}{2 \times 81}=61;\ \frac{9849}{3 \times 49}=67;$$

$$\frac{994}{14}=71;\ \frac{9928}{8 \times 17}=73;\ \frac{9954}{7 \times 18}=79;\ \frac{996}{12}=83;\ \frac{9968}{7 \times 16}=89;\ \frac{9894}{6 \times 17}=97.$$

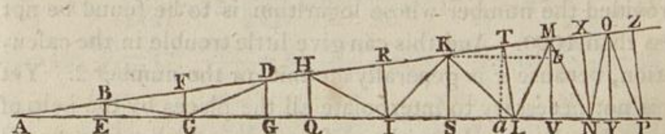
This quadrature of the hyperbola, and its application to the construction of logarithms, are still further explained by our celebrated author, in his treatise on Fluxions, published by Mr. Colson in 1736, where he gives all the three series for the areas AD, A*d*, B*d*, in general terms, the former the same as that published by Mercator, and the latter by Gregory; and he explains the manner of deriving the latter series from the former, namely by uniting together the two series for the spaces on each side of an ordinate, bounded by other ordinates at equal distances, every 2d term of each series is cancelled, and the result is a series converging much quicker than either of the former. And, in this treatise on fluxions, as well as in the letter before quoted, he recommends this as the most convenient way of raising a canon of logs. computing by the series the hyperbolic spaces answering to the prime numbers 2, 3, 5, 7, 11, &c, and dividing them by 2·3025850929940457, which is the area corresponding to the number 10, or else multiplying them by its reciprocal 0·4342944819032518, for the common logarithms. "Then the logarithms of all the numbers in the canon which are made by the multiplication of these, are to be found by the addition of their logarithms, as is usual. And the void places are to be interpolated afterwards by the help of this theorem: Let n be a number to which a logarithm is to be adapted, x the difference between that and the two nearest numbers equally distant on each side, whose logarithms are already found, and let d be half the difference of the logarithms; then the required logarithm of the number n will be obtained by adding $d + \frac{dx}{2n} + \frac{dx^3}{12n^3}$ &c to the logarithm of the less number." This theorem he demonstrates by the hyperbolic areas, and then proceeds thus; "The two first terms $d + \frac{dx}{2n}$ of this series I think to be accurate enough for the construction of a canon of logarithms, even though they were to be produced to 14 or 15 figures; provided the number whose logarithm is to be found be not less than 1000. And this can give little trouble in the calculation, because x is generally an unit, or the number 2. Yet it is not necessary to interpolate all the places by the help of this rule. For the logarithms of numbers which are produced by the multiplication or division of the number last found, may be obtained by the numbers whose logarithms were had before, by the addition or subtraction of their logarithms.—Moreover, by the differences of the logarithms, and by their 2d and 3d differences, if there be occasion, the void places may be more expeditiously supplied; the foregoing rule being to be applied only when the continuation of some full places is wanted, in order to obtain those differences, &c." So that Sir I. Newton of himself discovered all the series for the above quadrature, which were found out, and afterwards published, partly by Mercator and partly by Gregory; and these we may here exhibit in one view all together, and that in a general manner for any hyperbola, namely putting CA $= a$, AF $= b$, and AB $=$ A$b = x$; then will BD $= \frac{ab}{a+x}$, and $bd = \frac{ab}{a-x}$; whence the areas are as below, viz.

$$\text{AD} = bx - \frac{bx^2}{2a} + \frac{bx^3}{3a^2} - \frac{bx^4}{4a^3} + \frac{bx^5}{5a^4} \text{ \&c.}$$

$$\text{A}d = bx + \frac{bx^2}{2a} + \frac{bx^3}{3a^2} + \frac{bx^4}{4a^3} + \frac{bx^5}{5a^4} \text{ \&c.}$$

$$\text{B}d = 2bx + \frac{2bx^3}{3a^2} + \frac{2bx^5}{5a^4} + \frac{2bx^7}{7a^6} + \frac{2bx^9}{9a^8} \text{ \&c.}$$

In the same letter also, above quoted, to Mr. Oldenburg, our illustrious author teaches a method of constructing the trigonometrical canon of sines, by an easier method of multiple angles than that before delivered by Briggs, for the same purpose, because that in Sir Isaac's way radius or 1 is the first term, and double the sine or cosine of the first given angle is the 2d term, of all the proportions, by which the several successive multiple sines or cosines are found. The substance of the method is thus: The best foundation for the construction of the table of sines, is the continual addition of a given angle to itself, or to another given angle. As, if the angle A be to be added;



inscribe HI, IK, KL, LM, MN, NO, OP, &c, each equal to the radius AB; and to the opposite sides draw the perpendiculars BE, HQ, IR, KS, LT, MV, NX, OY, &c; so shall the angle A be the common difference of the angles HIQ, IKH, KLI, LMK, &c; their sines HQ, IR, KS, &c; and their cosines IQ, KR, LS, &c. Now let any one of them LMK, be given, then the rest will be thus found: Draw Ta and Kb perpendicular to SV and MV;

now because of the equiangular triangles ABE, TLa, KMb, ALT, AMV, &c, it will be AB : AE :: KT : Sa $(=\frac{1}{2}\text{LV}+\frac{1}{2}\text{LS})$ :: LT : Ta $(=\frac{1}{2}\text{MV}+\frac{1}{2}\text{KS},)$ and AB : BE :: LT : La $(=\frac{1}{2}\text{LS}-\frac{1}{2}\text{LV})$ :: KT $(=\frac{1}{2}\text{KM})$: $\frac{1}{2}$Mb $(=\frac{1}{2}\text{MV}-\frac{1}{2}\text{KS}.)$ Hence are given the sines and cosines KS, MV, LS, LV. And the method of continuing the progressions is evident. Namely,

$$\text{as AB : 2AE ::}\left\{\begin{array}{l}\text{LV : MT + MX :: MX : NV + NY \&c,}\\ \text{MV : NX + LT :: NX : OY + MV \&c,}\end{array}\right.$$

$$\text{or AB : 2BE ::}\left\{\begin{array}{l}\text{LV : NX − LT :: MX : OY − MV \&c,}\\ \text{MV : MT − MX :: NX : NV − NY \&c.}\end{array}\right.$$

And, on the other hand, AB : 2AE :: LS : KT + KR &c. Therefore put AB = 1, and make BE × LT = La, AE × KT = Sa, Sa − La = LV, 2AE × LV − TM = MX, &c.

The sense of these general theorems is this, that if p be any one among a series of angles in arithmetical progression, the angle d being their common difference, then as radius or

$$1 : 2\cos.\, d :: \left\{\begin{array}{l}\cos.\, p : \cos.\, p+d + \cos.\, p-d,\\ \sin.\, p : \sin.\, p+d + \sin.\, p-d,\end{array}\right.$$

$$1 : 2\sin.\, d :: \left\{\begin{array}{l}\cos.\, p : \sin.\, p+d - \sin.\, p-d,\\ \sin.\, p : \cos.\, p+d - \cos.\, p-d;\end{array}\right.$$

where the 4th terms of these proportions are the sums or differences of the sines or cosines of the two angles next less and greater than any angle p in the series; and therefore, subtracting the less extreme from the sum, or adding it to the difference, the result will be the greater extreme, or the next sine or cosine beyond that of the term p. And in the same manner are all the rest to be found. This method, it is evident, is equally applicable, whether the common difference d, or angle A, be equal to one term of the series or not: when it *is* one of the terms, then the whole series of sines and cosines becomes thus, viz, as 1 : 2 cos. d ::

$$\sin.\, d : \sin.\, 2d :: \sin.\, 2d : \sin.\, d + \sin.\, 3d :: \sin.\, 3d : \sin.\, 2d + \sin.\, 4d \ \&c.$$
$$\cos.\, d : 1 + \cos.\, 2d :: \cos.\, 2d : \cos.\, d + \cos.\, 3d :: \cos.\, 3d : \cos.\, 2d + \cos.\, 4d \ \&c.$$

which is the very method contained in the directions given by Abraham Sharp, for constructing the canon of sines.

Sir I. Newton remarks, that it only remains to find the sine and cosine of a first angle A, by some other method; and for this purpose, he directs to make use of some of his own infinite series: thus, by them will be found 1·57079&c for the quadrantal arc, the square of which is 2·4694&c; divide this square by the square of the number expressing the ratio of 90 degrees to the angle A, calling the quotient z; then 3 or 4 terms of this series $1 - \frac{z}{2} + \frac{z^2}{24} - \frac{z^3}{720} + \frac{z^4}{40320}$ &c, will give the cosine of that angle A. Thus we may first find an angle of 5 degrees, and thence the table be computed to the series of every 5 degrees; then these interpolated to degrees or half degrees by the same method, and these interpolated again; and so on as far as necessary. But two-thirds of the table being computed in this manner, the remaining third will be found by addition or subtraction only, as is well known.

Various other improvements in logarithms and trigonometry are owing to the same excellent personage; such as, the series for expressing the relation between circular arcs and their sines, cosines, versed-sines, tangents, &c; namely, the arc being a, the sine s, the versed-sine v, cosine c, tangent t, radius 1, then is

$$a = s + \tfrac{1}{6}s^3 + \tfrac{3}{40}s^5 + \tfrac{5}{112}s^7 + \tfrac{35}{1152}s^9 + \&c.$$
$$a = v^{\frac{1}{2}} + \tfrac{1}{6}v^{\frac{3}{2}} + \tfrac{3}{40}v^{\frac{5}{2}} + \tfrac{5}{112}v^{\frac{7}{2}} + \tfrac{35}{1152}v^{\frac{9}{2}} + \&c.$$
$$a = t - \tfrac{1}{3}t^3 + \tfrac{2}{5}t^5 - \tfrac{1}{7}t^7 + \tfrac{1}{9}t^9 - \&c.$$
$$s = a - \tfrac{1}{6}a^3 + \tfrac{1}{120}a^5 - \tfrac{1}{3040}a^7 + \tfrac{1}{362880}a^9 - \&c.$$
$$c = 1 - \tfrac{1}{2}a^2 + \tfrac{1}{24}a^4 - \tfrac{1}{720}a^6 + \tfrac{1}{40320}a^8 - \&c.$$
$$v = \tfrac{1}{2}a^2 - \tfrac{1}{24}a^4 + \tfrac{1}{720}a^6 - \tfrac{1}{40320}a^8 + \tfrac{1}{3628800}a^{10} - \&c.$$
$$t = a + \tfrac{1}{3}a^3 + \tfrac{2}{15}a^5 + \tfrac{17}{315}a^7 + \tfrac{62}{2835}a^9 + \&c.$$

Of Dr. Halley's Method.

Many other improvements in the construction of logarithms are also derived from the same doctrine of fluxions, as we shall show hereafter. In the mean time proceed we to the ingenious method of the learned Dr. Edmund Halley, secretary to the Royal Society, and the second astronomer royal, having succeeded Mr. Flamsteed in that honourable office in the year 1719, at the Royal Observatory at Greenwich, where he died the 14th January 1742, in the 86th year of his age. His method was first printed in the Philosophical Transactions for the year 1695, and it is entitled "A most compendious and facile method for constructing the logarithms, exemplified and demonstrated from the nature of numbers, without any regard to the hyperbola, with a speedy method for finding the number from the given logarithm."

Instead of the more ordinary definition of logarithms, as *numerorum proportionalium æquidifferentes comites*, in this tract our learned author adopts this other, *numeri rationem exponentes*, as being better adapted to the principle on which logarithms are here constructed, where those quantities are not considered as the logarithms of the numbers, for example, of 2, or of 3, or of 10, but as the logarithms of the ratios of 1 to 2, or 1 to 3, or 1 to 10. In this consideration he first pursues the idea of Kepler and Mercator, remarking that any such ratio is proportional to, and is measured by, the number of equal ratiunculæ contained in each; which ratiunculæ are to be understood as in a continued scale of proportionals, infinite in number, between the two terms of the ratio; which infinite number of mean proportionals, is to that infinite number of the like and equal ratiunculæ between any other two terms, as the logarithm of the one ratio, is to the logarithm of the other: thus, if there be supposed between 1 and 10 an infinite scale of mean proportionals, whose number is 100000 &c in infinitum; then between 1 and 2 there will be 30102 &c of such proportionals; and between 1 and 3 there will be 47712 &c of them; which numbers therefore are the logarithms of the ratios of 1 to 10, 1 to 2, and 1 to 3. But for the sake of *his* mode of constructing logarithms, he changes this idea of *equal* ratiunculæ, for that of other ratiunculæ, so constituted, as that the *same* infinite number of them shall be contained in the ratio of 1 to every other number whatever; and that therefore these latter ratiunculæ will be of *unequal* or different magnitudes in all the different ratios, and in such sort, that in any one ratio, the *magnitude* of each of the ratiunculæ in this latter case, will be as the *number* of them in the former. And therefore, if between 1 and any number proposed, there be taken any infinity of mean proportionals, the infinitely small augment or decrement of the first of those means from the first term 1, will be a ratiuncula of the ratio of 1 to the said number; and as the number of all the ratiunculæ in these continued proportionals is the same, their sum, or the whole ratio, will be directly proportional to the magnitude of one of the said ratiunculæ in each ratio. But it is also evident that the first of any number of means, between 1 and any number, is always equal to such root of that number, whose index is expressed by the number of those proportionals from 1: so, if m denote the number of proportionals from 1, then the first term after 1 will be the mth root of that number. Hence, the indefinite root of any number being extracted, the differentiola of the said root from unity, shall be as the logarithm of that number. So if there be required the log. of the ratio of 1 to $1 + q$; the first term after 1 will be $(1 + q)^{\frac{1}{m}}$, and theref. the required log. will be as $(1 + q)^{\frac{1}{m}} - 1$. But, $(1 + q)^{\frac{1}{m}}$ is $= 1 + \frac{1}{m}q + \frac{1}{m} \cdot \frac{1-m}{2m}q^2 + \frac{1}{m} \cdot \frac{1-m}{2m} \cdot \frac{1-2m}{3m}q^3$ &c; or by omitting the 1 in the compound numerators, as infinitely small in respect of the infinite number m, the same series will become $1 + \frac{1}{m}q + \frac{1}{m} \cdot \frac{-m}{2m}q^2 + \frac{1}{m} \cdot \frac{-m}{2m} \cdot \frac{-2m}{3m}q^3$ &c, or by abbreviation it is $1 + \frac{1}{m}q - \frac{1}{2m}q^2 + \frac{1}{3m}q^3 - \frac{1}{4m}q^4$ &c; and hence, finding the differentiola by subtracting 1, the logarithm of the ratio of 1 to $1 + q$ is as $\frac{1}{m} \times (q - \frac{1}{2}q^2 + \frac{1}{3}q^3 - \frac{1}{4}q^4 + \frac{1}{5}q^5 - \frac{1}{6}q^6$ &c.) Now the index m may be taken equal to any infinite number, and thus all the varieties of scales of logarithms may be produced: so, if m be taken 1000000&c, the theorem will give Napier's logarithms; but if m be taken equal to 230258&c, there will arise Briggs's logarithms.

This theorem being for the increasing ratio of 1 to $1 + q$: if that for the decreasing ratio of 1 to $1 - q$ be also sought, it will be obtained by a proper change of the signs, by which the decrement of the first of the infinite number of proportionals, will be found to be $\frac{1}{m}$ into $q + \frac{1}{2}q^2 + \frac{1}{3}q^3 + \frac{1}{4}q^4$ &c, which therefore is as the logarithm of the ratio of 1 to $1 - q$.

Hence the terms of any ratio being a and b, q becomes $\frac{b-a}{a}$, or the difference divided by the less term, when it is an increasing ratio; or $q = \frac{b-a}{b}$ when the ratio is decreasing, or as b to a. Therefore the logarithm of the same ratio may be doubly expressed; for, putting x for the difference $b - a$ of the terms, it will be

$$\text{either } \frac{1}{m} \text{ into } \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \&c.$$

$$\text{or } \frac{1}{m} \text{ into } \frac{x}{b} + \frac{x^2}{2b^2} + \frac{x^3}{3b^3} + \frac{x^4}{4b^4} + \&c.$$

But if the ratio of a to b be supposed divided into two parts, namely, into the ratio of a to $\frac{1}{2}a + \frac{1}{2}b$ or $\frac{1}{2}z$, and the ratio of $\frac{1}{2}z$ to b, then will the sum of the logarithms of those two ratios be the logarithm of the ratio of a to b. Now by substituting in the foregoing series, the logarithms of those two ratios will

$$\text{be } \frac{1}{m} \text{ into } \frac{x}{z} + \frac{x^2}{2z^2} + \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} \&c.$$

$$\text{and } \frac{1}{m} \text{ into } \frac{x}{z} - \frac{x^2}{2z^2} + \frac{x^3}{3z^3} - \frac{x^4}{4z^4} + \frac{x^5}{5z^5} \&c; \text{ and hence the sum,}$$

$$\text{or } \frac{1}{m} \text{ into } \frac{2x}{z} + \frac{2x^3}{3z^3} + \frac{2x^5}{5z^5} + \frac{2x^7}{7z^7} + \frac{2x^9}{9z^9} + \&c,$$

will be the logarithm of the ratio of a to b.

Further, if from the logarithm of the ratio of a to $\frac{1}{2}z$, be taken that of $\frac{1}{2}z$ to b, we shall have the logarithm of the ratio of ab to $\frac{1}{4}z^2$; and the half of this gives that of $\sqrt{ab}$ to $\frac{1}{2}z$, or of the geometrical mean to the arithmetical mean. And consequently the logarithm of this ratio will be equal to half the difference of that of the above two ratios, and will therefore be $\frac{1}{m}$ into $\frac{x^2}{2z^2} + \frac{x^4}{4z^4} + \frac{x^6}{6z^6} + \frac{x^8}{8z^8} + \&c.$

The above series are similar to some that were before given by Newton and Gregory, for the same purpose, deduced from the consideration of the hyperbola. But the rule which is properly our author's own, is that which follows, and is derived from the series above given for the logarithm of the sum of two ratios. For the ratio of ab to $\frac{1}{4}z^2$ or $\frac{1}{4}a^2 + \frac{1}{2}ab + \frac{1}{4}b^2$, having the difference of its terms $\frac{1}{4}a^2 - \frac{1}{2}ab + \frac{1}{4}b^2$ or $(\frac{1}{2}b - \frac{1}{2}a)^2$ or $\frac{1}{4}x^2$, which in the case of finding the logs. of prime numbers is always 1, if we call the sum of the terms $\frac{1}{4}z^2 + ab = y^2$, the log. of the ra. of $\surd ab$ to $\frac{1}{2}a+\frac{1}{2}b$ or $\frac{1}{2}z$ will be found to be

$$\frac{1}{m} \text{ into } \frac{1}{y^2} + \frac{1}{3y^6} + \frac{1}{5y^{10}} + \frac{1}{7y^{14}} + \frac{1}{9y^{18}} + \&c.$$

And these rules our learned author exemplifies by some cases in numbers, to show the easiest mode of application in practice.

Again, by means of the same binomial theorem he resolves, with equal facility, the reverse of the problem, namely, from the log. given, to find its number or ratio: For, as the log. of the ratio of 1 to $1+q$ was proved to be $(1+q)^{\frac{1}{m}}-1$, and that of the ratio of 1 to $1-q$ to be $1-(1-q)^{\frac{1}{m}}$; hence, calling the given logarithm L, in the former case it will be $(1+q)^{\frac{1}{m}} = 1 + \text{L}$, and in the latter $(1-q)^{\frac{1}{m}} = 1 - \text{L}$; and therefore $1+q=(1+\text{L})^m$ and $1-q=(1-\text{L})^m$ } that is, by the binomial theorem,

$$1+q=1+m\text{L}+\tfrac{1}{2}m^2\text{L}^2+\tfrac{1}{6}m^3\text{L}^3+\tfrac{1}{24}m^4\text{L}^4+\tfrac{1}{120}m^5\text{L}^5 \ \&c,$$

and $1-q=1-m\text{L}+\tfrac{1}{2}m^2\text{L}^2-\tfrac{1}{6}m^3\text{L}^3+\tfrac{1}{24}m^4\text{L}^4-\tfrac{1}{120}m^5\text{L}^5$ &c, m being any infinite index whatever, differing according to the scale of logarithms, being 1000&c in Napier's or the hyperbolic logarithms, and 2302585&c in Briggs's.

If one term of the ratio, of which L is the logarithm, be given, the other term will be easily obtained by the same rule: For if L be Napier's logarithm, of the ratio of a the less term, to b the greater, then, according as a or b is given, we shall have,

$$b = a \text{ into } 1 + \text{L} + \tfrac{1}{2}\text{L}^2 + \tfrac{1}{6}\text{L}^3 + \tfrac{1}{24}\text{L}^4 + \&c,$$

$$a = b \text{ into } 1 - \text{L} + \tfrac{1}{2}\text{L}^2 - \tfrac{1}{6}\text{L}^3 + \tfrac{1}{24}\text{L}^4 - \&c.$$

Hence, by help of the logarithms contained in the tables, may easily be found the number to any given log. to a great extent. For if the small difference between the given log. L and the nearest tabular logarithm, either greater or less, be called l, and the number answering to the tabular logarithm a, when it is less than the given logarithm, but b when greater; it will follow, that the number answering to the log. L, will be

$$\text{either } a \text{ into } 1 + l + \tfrac{1}{2}l^2 + \tfrac{1}{6}l^3 + \tfrac{1}{24}l^4 + \tfrac{1}{120}l^5 + \&c,$$

$$\text{or } b \text{ into } 1 - l + \tfrac{1}{2}l^2 - \tfrac{1}{6}l^3 + \tfrac{1}{24}l^4 - \tfrac{1}{120}l^5 + \&c,$$

which series converge so quickly, l being always very small, that the first two terms $1 \pm l$ are generally sufficient to find the number to 10 places of figures.

Dr. Halley subjoins also an easy approximation for these series; by which it appears, that the number answering to the log. is nearly $\frac{1+\frac{1}{2}l}{1-\frac{1}{2}l} \times a$ or $\frac{1-\frac{1}{2}l}{1+\frac{1}{2}l} \times b$ in Napier's logs.; and $\frac{n+\frac{1}{2}l}{n-\frac{1}{2}l} \times a$ or $\frac{n-\frac{1}{2}l}{n+\frac{1}{2}l} \times b$ in Briggs's logarithms; where n is $=$ $\cdot 434294481903\&c = \frac{1}{m}$.

Of Mr. Sharp's Methods.

The labours of Mr. Abraham Sharp, of Little-Horton, near Bradford in Yorkshire, in this branch of mathematics, were very great and meritorious. His merit however consisted rather in the improvement and illustration of the methods of former writers, than in the invention of any new ones of his own. In this way he greatly extended and improved Dr. Halley's method, above described, as also those of Mercator and Wallis; illustrating these improvements by extensive calculations, and by them computing table 5 of my collection of Mathematical Tables, consisting of the logarithms of all numbers to 100, and of all prime numbers to 1100, each to 61 places. He also composed a neat compendium of the best methods for computing the natural sines, tangents, and secants, chiefly from the rules before given by Newton; and by Newton's or Gregory's series $a = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7$ &c, for the arc in terms of the tangent, he computed the circumference of the circle to 72 places, namely from the arc of 30 degrees, whose tangent t is $= \sqrt{\frac{1}{3}}$ to the radius 1. Other surprizing instances of his industry and labour appear in his Geometry Improv'd, printed in 1717, and signed A. S. Philomath, from which the 5th table of logarithms above-mentioned was extracted. This ingenious man was sometime assistant at the Royal Observatory to Mr. Flamsteed the first astronomer royal; and, being one of the most accurate and indefatigable computers that ever existed, he was for many years the common resource for Mr. Flamsteed, Sir Jonas Moore, Dr. Halley, &c, in all intricate and troublesome calculations. He afterwards retired to his native place at Little-Horton, where, after a life spent in intense study and calculations, he died the 18th July 1742, in the 91st year of his age.

Of the Construction of Logarithms by Fluxions.

It appears by the very definition and description given by Napier of his logarithms, as stated in page 341 of this vol. that the fluxion of his, or the hyperbolic logarithm, of any number, is a fourth proportional to that number, its logarithm, and unity; or, which is the same thing, that it is equal to the fluxion of the number divided by the number: For the description shows, that $z1 : za$ or $1 :: \dot{z1}$ the fluxion of $za : \dot{za}$, which therefore is $= \frac{\dot{z1}}{z1}$; but $\dot{za}$ is also equal to the fluxion of the logarithm A&c, by the description; therefore the fluxion of the logarithm is equal to $\frac{\dot{z1}}{z1}$, the fluxion of the quantity divided by the quantity itself. The same thing appears again at art. 2 of that little piece, in the appendix to his Constructio Logarithmorum, entitled Habitudines Logarithmorum et suorum naturalium numerorum invicem, where he observes that, as any greater quantity is to a less, so is the velocity of the increment or decrement of the logarithms at the place of the less quantity, to that at the greater. Now this velocity of the increment or decrement of the logarithms being the same thing as their fluxions, that proportion is this, $x : a ::$ flux. log. a : flux. log. x; hence if a be $= 1$, as at the beginning of the table of numbers, where the fluxion of the logs. is the index or characteristic c, which is also 1 in Napier's or the hyperbolic logarithms, and 43429&c in Briggs's, the same proportion becomes $x : 1 :: c :$ flux. log. x; but the constant fluxion of the numbers is also 1, and therefore that proportion is also this, $x : \dot{x} :: c : \frac{c\dot{x}}{x} =$ the fluxion of the log. of x; and in the hyperbolic logs. where c is $= 1$, it becomes $\frac{\dot{x}}{x} =$ the fluxion of Napier's or the hyperbolic logarithm of

x. This same property has also been noticed by many other authors since Napier's time. And the same, or a similar property, is evidently true in all systems of logarithms whatever, namely, that the modulus of the system is to any number, as the fluxion of its logarithm is to the fluxion of the number.

Now from this property, by means of the doctrine of fluxions, are derived other ways for making logarithms, which have been illustrated by many writers on this branch, as Craig, John Bernoulli, and almost all the writers on fluxions. And this method chiefly consists in expanding the reciprocal of the given quantity in an infinite series, then multiplying each term by the fluxion of the said quantity, and lastly taking the fluents of the terms; by which there arises an infinite series of terms for the logarithm sought. So, to find the logarithm of any number N; put any compound quantity for N, as suppose $\frac{n+x}{n}$;

then the flux. of the log. or $\frac{\dot{N}}{N}$ being $\frac{\dot{x}}{n+x} = \frac{\dot{x}}{n} - \frac{x\dot{x}}{nn} + \frac{x^2\dot{x}}{n^3} - \frac{x^3\dot{x}}{n^4}$ &c,

the fluents give log. of N or log. of $\frac{n+x}{n} = \frac{x}{n} - \frac{x^2}{2n^2} + \frac{x^3}{3n^3} - \frac{x^4}{4n^4}$ &c.

And writing $-x$ for x gives log. $\frac{n-x}{n} = -\frac{x}{n} - \frac{x^2}{2n^2} - \frac{x^3}{3n^3} - \frac{x^4}{4n^4}$ &c.

Also, because $\frac{n}{n \pm x} = 1 \div \frac{n \pm x}{n}$, or log. $\frac{n}{n \pm x} = 0 - \log. \frac{n \pm x}{n}$

$$\text{theref. log. } \frac{n}{n+x} = -\frac{x}{n} + \frac{x^2}{2n^2} - \frac{x^3}{3n^3} + \frac{x^4}{4n^4} \text{ \&c,}$$

$$\text{and log. } \frac{n}{n-x} = +\frac{x}{n} + \frac{x^2}{2n^2} + \frac{x^3}{3n^3} + \frac{x^4}{4n^4} \text{ \&c.}$$

And by adding and subtracting any of these series, to or from one another, and multiplying or dividing their corresponding numbers, various other series for logarithms may be found, converging much quicker than these do.

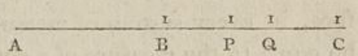
In like manner, by assuming quantities otherwise compounded, for the value of N, various other forms of logarithmic series may be found by the same means.

Of Mr. Cotes's Logometria.

Mr. Roger Cotes was elected the first Plumian professor of astronomy and experimental philosophy in the university of

Cambridge, January 1706, which appointment he filled with the greatest credit, till he died the 5th of June 1716, in the prime of life, having not quite completed the 34th year of his age. His early death was a great loss to the mathematical world, as his genius and abilities were of the brightest order, as is manifest by the specimens of his performance given to the public. Among these is, his Logometria, first printed in number 338 of the Philosophical Transactions, and afterwards in his Harmonia Mensurarum, published in 1722, with his other works, by his relation and successor, in the Plumian professorship, Dr. Robert Smith. In this piece he first treats, in a general way, of measures of ratios, which measures, he observes, are quantities of any kind, whose magnitudes are analogous to the magnitudes of the ratios, these magnitudes mutually increasing and decreasing together in the same proportion. He remarks, that the ratio of equality has no magnitude, because it produces no change by adding and subtracting; that the ratios of greater and less inequality, are of different affections; and therefore if the measure of the one of these be considered as positive, that of the other will be negative; and the measure of the ratio of equality nothing: That there are endless systems of these, which have all their measures of the same ratios proportional to certain given quantities, called *moduli*, which he defines afterwards, and the ratio of which they are the measures, each in its peculiar system, is called the modular ratio, *ratio modularis*, which ratio is the same in all systems. He then adverts to logarithms, which he considers as the numerical measures of ratios, and he describes the method of arranging them in tables, with their uses in multiplication and division, raising of powers and extracting of roots, by means of the corresponding operations of addition and subtraction, multiplication and division.

After this introduction, which is only a slight abridgment of the doctrine long before very amply treated of by others, and particularly by Kepler and Mercator, we arrive at the first proposition, which has justly been censured as obscure and imperfect, seemingly through an affectation of brevity, intricacy, and originality, without sufficient room for a display of this quality. The reasoning in this proposition, such as it is, seems to be something between that of Kepler and the principles of fluxions, to which the quantities and expressions are nearly allied. However, as it is my duty rather to narrate than explain, I shall here exhibit it exactly as it stands. This proposition is, to determine the measure of any ratio, as for instance that of AC to AB, and which is effected in this manner: Conceive the difference BC to be divided into innumerable very small particles, as PQ, and the ratio between AC and AB into as many such very small ratios, as between AQ and AP: then if the magnitude of the ratio between AQ and AP be given, by dividing, there will also be given that of PQ to AP; and therefore, this being given, the magnitude of the ratio between AQ and AP may be expounded by the given quantity $\frac{PQ}{AP}$; for, AP remaining constant, conceive the particle PQ to be augmented or diminished in any proportion, and in the same proportion will the magnitude of the ratio between AQ and AP be augmented or diminished: Also, taking any determinate quantity M, the same may be expounded by $M \times \frac{PQ}{AP}$; and therefore the quantity $M \times \frac{PQ}{AP}$ will be the measure of the ratio between AQ and AP. And this measure will have divers magnitudes, and be accommodated to divers systems, according to the divers magnitudes of the assumed quantity M, which therefore is called the *modulus* of the system. Now, like as the sum of all the ratios AQ to AP is equal to the proposed ratio AC to AB, so the sum of all the measures $M \times \frac{PQ}{AP}$, found by the known methods, will be equal to the required measure of the said proposed ratio.



The general solution being thus dispatched, from the general expression, Cotes next deduces other forms of the measure, in several corollaries and scholia: as 1st, the terms AP, AQ, approach the nearer to equality as the small difference PQ is less; so that either $M \times \frac{PQ}{AP}$ or $M \times \frac{PQ}{AQ}$ will be the measure of the ratio between AQ and AP, to the modulus M. 2d, That hence the modulus M, is to the measure of the ratio between AQ and AP, as either AP or AQ is to their difference PQ. 3d, The ratio between AC and AB being given, the sum of all the $\frac{PQ}{AP}$ will be given; and the sum of all the $M \times \frac{PQ}{AP}$ is as M: therefore the measure of any given ratio, is as the modulus of the system from which it is taken. 4th, Therefore, in every system of measures, the modulus will always be equal to the measure of a certain determinate and immutable ratio; which therefore he calls the modular ratio. 5th, To illustrate the solution by an example: let z be any determinate and permanent quantity, x a variable or indeterminate quantity, and $\dot{x}$ its fluxion; then, to find the measure of the ratio between $z + x$ and $z - x$, put this ratio equal to the ratio between y and 1, expounding the number y by AP, its fluxion $\dot{y}$ by PQ, and 1 by AB: then the fluxion of the required measure of the ratio between y and 1 is $M \times \frac{\dot{y}}{y}$. Now, for y, restore its val. $\frac{z+x}{z-x}$, and for $\dot{y}$ the flux. of that val. $\frac{2z\dot{x}}{(z-x)^2}$, so shall the flux. of the measure become $2M \times \frac{z\dot{x}}{zz-xx}$, or $2M$ into $\frac{\dot{x}}{z} + \frac{\dot{x}x^2}{z^3} + \frac{\dot{x}x^4}{z^5} + \&c$; and therefore that measure will be $2M$ into $\frac{x}{z} + \frac{x^3}{3z^3} + \frac{x^5}{5z^5} + \&c$. In like manner the measure of the ratio between $1 + v$ and 1, will be found to be - - - - M into $v - \frac{1}{2}v^2 + \frac{1}{3}v^3 - \frac{1}{4}v^4 + \&c$. And hence, to find the number from the logarithm given, he reverts the series in this manner: If the last measure be called m, we

shall have $\frac{m}{M}$ or $Q = v - \frac{1}{2}v + \frac{1}{3}v^3 - \frac{1}{4}v^4 + \frac{1}{5}v^5 \ \&c,$

therefore $Q^2 =$ - $v^2 - v^3 + \frac{11}{12}v^4 - \frac{5}{6}v^5 \ \&c,$

and $Q^3 =$ - - - $v^3 - \frac{3}{2}v^4 + \frac{7}{4}v^5 \ \&c,$

and $Q^4 =$ - - - - - $v^4 - 2v^5 \ \&c,$

and $Q^5 =$ - - - - - - - $v^5 \ \&c;$

then, by adding continually, we shall have,

$$Q + \tfrac{1}{2}Q^2 = v - \tfrac{1}{6}v^3 + \tfrac{5}{24}v^4 - \tfrac{13}{60}v^5 \text{ \&c},$$
$$Q + \tfrac{1}{2}Q^2 + \tfrac{1}{6}Q^3 = v - \tfrac{1}{24}v^4 + \tfrac{3}{40}v^5 \text{ \&c},$$
$$Q + \tfrac{1}{2}Q^2 + \tfrac{1}{6}Q^3 + \tfrac{1}{24}Q^4 = v - \tfrac{1}{120}v^5 \text{ \&c},$$
$$Q + \tfrac{1}{2}Q^2 + \tfrac{1}{6}Q^3 + \tfrac{1}{24}Q^4 + \tfrac{1}{120}Q^5 = v \text{ \&c},$$

that is $v = Q + \frac{1}{2}Q^2 + \frac{1}{6}Q^3 + \frac{1}{24}Q^4 + \frac{1}{120}Q^5$ &c. And therefore the required ratio of $1 + v$ to 1, is equal to the ratio of $1 + Q + \frac{1}{2}Q^2$ &c to 1. Now put $m = M$, or $Q = 1$, and the above will become the ratio of $1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120}$ &c to 1, for the constant modular ratio. In like manner, if the ratio between 1 and $1 - v$ be proposed, the measure of this ratio will come out M into $v + \frac{1}{2}v^2 + \frac{1}{3}v^3 + \frac{1}{4}v^4$ &c; which being called m, and $\frac{m}{M} = Q$, that ratio will be the ratio of 1 to $1 - Q + \frac{1}{2}Q^2 - \frac{1}{6}Q^3 + \frac{1}{24}Q^4$ &c. And hence, taking $m = M$, or $Q = 1$, the said modular ratio will also be the ratio of 1 to $1 - \frac{1}{1} + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}$ &c. And the former of these expressions, for the modular ratio, comes out the ratio of 2.718281828459 &c to 1, and the latter the ratio of 1 to 0.367879441171 &c, which number is the reciprocal of the former.

In the 2d prop. the learned author gives directions for constructing Briggs's canon of logarithms, namely, first by the general series 2M into $\frac{x}{z} + \frac{x^3}{3z^3} + \frac{x^5}{5z^5} +$ &c, finding the logarithms of a few such ratios as that of 126 to 125, 225 to 224, 2401 to 2400, 4375 to 4374, &c, from which the logarithm of 10 will be found to be 2.302585092994 &c, when M is 1; but since Briggs's log. of 10 is 1, therefore as 2.302585 &c is to the mod. 1, so is 1 (Briggs's log. of 10) to 0.434294481903&c, which therefore is the modulus of Briggs's logarithms. Hence he deduces the logarithms of 7, 5, 3, and 2. In like manner are the logarithms of other prime numbers to be found, and from them the logarithms of composite numbers by addition and subtraction only.

Cotes then remarks, that the first term of the general series 2M into $\frac{x}{z} + \frac{x^3}{3z^3} + \frac{x^5}{5z^5} +$ &c, will be sufficient for the logarithms of intermediate numbers between those in the table, or even for numbers beyond the limits of the table. Thus, to find the logarithm answering to any intermediate number; let a and e be two numbers, the one the given number, and the other the nearest tabular number, a being the greater, and e the less of them; put $z = a + e$ their sum, $x = a - e$ their difference, $\lambda =$ the logarithm of the ratio of a to e, that is the excess of the logarithm of a above that of e: so shall the said difference of their logarithms be $\lambda = 2\text{M} \times \frac{x}{z}$ very nearly. And, if there be required the number answering to any given intermediate logarithm, because λ is $= \frac{2\text{M}x}{z} = \frac{2\text{M}x}{2a-x}$ or $\frac{2\text{M}x}{2e+x}$, theref. $x = \frac{\lambda a}{\text{M}+\frac{1}{2}\lambda}$ or $\frac{\lambda e}{\text{M}-\frac{1}{2}\lambda}$ very nearly.

In the 3d prop. the ingenious author teaches how to convert the canon of logarithms into logarithms of any other system, by means of their *moduli*. And, in several more propositions, he exemplifies the canon of logarithms in the solution of various important problems in geometry and physics; such as the quadrature of the hyperbola, the description of the logistica, the equiangular spiral, the nautical meridian, &c, the descent of bodies in resisting mediums, the density of the atmosphere at any altitude, &c, &c.

Of Dr. Taylor's Construction of Logarithms.

Dr. Brook Taylor, a very learned mathematician, and secretary to the Royal Society, who died at Somerset-house, Nov. 1731, gave the following method of constructing logarithms, in number 352 of the Philosophical Transactions. His method is founded on these three considerations: 1st, that the sum of the logarithms of any two numbers, is the logarithm of the product of those numbers; 2d, that the logarithm of 1 is nothing, and consequently that the nearer any number is to 1, the nearer will its logarithm be to 0; 3d, that the product of two numbers or factors, of which the one is greater and the other less than 1, is nearer to 1 than that factor is which is on the same side of 1 with itself; so of the two numbers $\frac{2}{3}$ and $\frac{4}{3}$, the product $\frac{8}{9}$ is less than 1, but yet nearer to it than $\frac{2}{3}$ is, which is also less than 1. On these principles he founds the present approximation, which he explains by the following example.

To find the relation between the logs. of 2 and 10: In order to this, he assumes two fractions, as $\frac{128}{100}$ and $\frac{8}{10}$, or $\frac{2^7}{10^2}$, and $\frac{2^3}{10}$, whose numerators are powers of 2, and their denominators powers of 10, the one fraction being greater, and the other less than unity or 1. Having set these two down, in the form of decimal fractions, below each other in the first column of the following table, and in the second column A and B for their logarithms, expressing by an equation how they are

1,280000000000	A = . . =	7*l*2 −	2*l*10	*l*2 > 0,28
0,800000000000	B = . . =	3*l*2 −	*l*10	< 0,33
1,024000000000	C = A + B =	10*l*2 −	3*l*10	> 0,300
0,990352031429	D = B + 9C =	93*l*2 −	28*l*10	< 0,30107
1,004336277664	E = C + 2D =	169*l*2 −	59*l*10	> 0,301020
0,998959536107	F = D + 2E =	485*l*2 −	146*l*10	< 0,3010309
1,000162894165	G = E + 4F =	2136*l*2 −	643*l*10	> 0,30102996
0,999936281874	H = F + 6G =	13301*l*2 −	4004*l*10	< 0,301029997
1,000035441215	I = G + 2H =	28738*l*2 −	8651*l*10	> 0,3010299951
0,999971720830	K = H + I =	42039*l*2 −	12655*l*10	< 0,3010299959
1,000007161046	L = I + K =	70777*l*2 −	21306*l*10	> 0,30102999562
0,999993203514	M = K + 3L =	254370*l*2 −	76573*l*10	< 0,30102999567
1,000000364511	N = L + M =	325147*l*2 −	97879*l*10	> 0,3010299956635
0,999999764687	O = M + 18N =	6107016*l*2 −	1838335*l*10	< 0,3010299956640
comp.ar.235313				
0 = 3645110 + 235313N =		2302585825187*l*2 −	693147400972*l*10	> 0,3010299956663987

composed of the logarithms of 2 and 10, the numbers in question, those logarithms being denoted thus, *l*2 and *l*10. Then multiplying the two numbers in the first column together, there is produced a third number 1,024, against which is written C, for its logarithm, expressing likewise by an equation in what manner $\bar{c}$ is formed of the foregoing logarithms A and B. And in the same manner the calculation is continued throughout; only observing this compendium, that before multiplying the two last numbers already entered in the table, to consider what power of one of them must be used to bring the product the nearest that can be to unity. Now after having continued the table a little way, this is found by only dividing the differences of the numbers from unity one by the other, and taking the nearest quotient for the index of the power sought. Thus, the second and third numbers in the table being 0,8 and 1,024, their differences from unity are 0,200 and 0,024; hence 0,200 ÷ 0,024 gives 9 for the index; and therefore multiplying the 9th power of 1,024 by 0,8, produces the next number 0,990352031429, whose logarithm is D = B + 9C.

When the calculation is continued in this manner till the numbers become small enough, or near enough to 1, the last logarithm is supposed equal to nothing, which gives an equation expressing the relation of the logarithms, and thence the required logarithm is determined. Thus, supposing G = 0, we have $2136l2 - 643l10 = 0$, and hence, because the logarithm of 10 is 1, we obtain $l2 = \frac{643}{2136} = 0,30102996$, too small in the last figure only; which so happens, because the number corresponding to G is greater than 1. And in this manner are all the numbers in the third or last column obtained, which are continual approximations to the logarithm of 2.

There is another expedient, which renders this calculation still shorter, and it is founded on this consideration: that when x is small, $(1+x)^n$ is nearly $= 1+nx$. Hence if $1+x$ and $1-z$ be the two last numbers already found in the first column of the table, the product of their powers $(1+x)^m \times (1-z)^n$ will be nearly $= 1$; and hence the relation of m and n may be thus found, $(1+x)^m \times (1-z)^n$ is nearly $= (1+mx)\times(1-nz) = 1+mx-nz-mnxz = 1+mx-nz$ nearly, which being also $= 1$ nearly, therefore $m : n :: z : x :: l.(1-z) : l.(1+x)$; whence $xl.(1-z) + zl.(1+x) = 0$. For example, let 1,024 and 0,990352 be the last numbers in the table, their logs. being C and D: here we have $1,024 = 1+x$, and $0,990352 = 1-z$; conseq. $x = 0,024$, and $z = 0,009648$, and hence the ratio $\frac{z}{x}$ in small numbers is $\frac{201}{500}$. So that, for finding the logarithms proposed, we may take $500D + 201C = 48510l2 - 14603l10 = 0$; which gives $l2 = 0,3010307$. And in this manner are found the numbers in the last line of the table.

Of Mr. Long's Method.

In number 339 of the Philosophical Transactions, are given a brief table and method for finding the logarithm to any number, and the number to any logarithm, by Mr. John Long, B. D. Fellow of C. C. C. Oxon. This table and method are similar to those described in chap. 14, of Briggs's Arith. Log. differing only in this, that in this table, by Mr. Long, the logarithms, in each class, are in arithmetical progression, the common difference being 1; but in Briggs's little table, the column of natural numbers has the like common difference. The table consists of eight classes of logarithms, and their corresponding numbers, as follow:

L.	Nat. Numb.	Log.	Nat. Numb.	Log.	Nat. Numb.	Log.	Nat. Numb.
,9	7,943282347	,009	1,020939484	,00009	1,000207254	,0000009	1,000002072
,8	6,309573445	8	1,018591388	8	1,000184224	8	1,000001842
,7	5,011872336	7	1,016248694	7	1,000161194	7	1,000001611
,6	3,981071706	6	1,013911386	6	1,000138165	6	1,000001381
,5	3,162277660	5	1,011579454	5	1,000115136	5	1,000001151
,4	2,511886432	4	1,009252886	4	1,000092106	4	1,000000921
,3	1,995262315	3	1,006931669	3	1,000069086	3	1,000000690
,2	1,584893193	2	1,004615794	2	1,000046053	2	1,000000460
,1	1,258925412	1	1,002305238	1	1,000023026	1	1,000000230
,09	1,230268771	,0009	1,002074475	,000009	1,000020724	,00000009	1,000000207
8	1,202264435	8	1,001843766	8	1,000018421	8	1,000000184
7	1,174897555	7	1,001613109	7	1,000016118	7	1,000000161
6	1,148153621	6	1,001382506	6	1,000013816	6	1,000000138
5	1,122018454	5	1,001151956	5	1,000011513	5	1,000000115
4	1,096478196	4	1,000921459	4	1,000009210	4	1,000000092
3	1,071519305	3	1,000691015	3	1,000006908	3	1,000000069
2	1,047128548	2	1,000460623	2	1,000004605	2	1,000000046
1	1,023292992	1	1,000230285	1	1,000002302	1	1,000000023

where, because the logarithms in each class are the continual multiples 1, 2, 3, &c, of the lowest, it is evident that the natural numbers are so many scales of geometrical proportionals, the lowest being the common ratio, or the ascending numbers are the 1, 2, 3, &c, powers of the lowest, as expressed by the figures 1, 2, 3, &c, of their corresponding logarithms. Also the last number in the first, second, third, &c class, is the 10th, 100th, 1000th, &c root of 10; and any number in any class, is the 10th power of the corresponding number in the next following class.

To find the logarithm of any number, as suppose of 2000, by this table, Look in the first class for the number next less than the first figure 2, and it is 1,995262315, against which is 3 for the first figure of the logarithm sought. Again, dividing 2, the number proposed, by 1,995262315, the number found in the table, the quotient is 1,002374467; which being looked for in the second class of the table, and finding neither its equal nor a less, 0 is therefore to be taken for the second figure of the logarithm; and the same quotient 1,002374467 being looked for in the third class, the next less is there found to be 1,002305238, against which is 1 for the third figure of the logarithm; and dividing the quotient 1,002374467 by the said next less number 1,002305238, the new quotient is 1,000069070; which being sought in the fourth class, gives 0, but sought in the fifth class gives 2, which are the fourth and fifth figures of the logarithm sought: again, dividing the last quotient by 1,000046053, the next less number in the table, the quotient is 1,000023015, which gives 9 in the 6th class for the 6th figure of the logarithm sought: and again dividing the last quotient by 1,000020724, the next less number, the quotient is 1,000002291, the next less than which, in the 7th class, gives 9 for the 7th figure of the logarithm: and dividing the last quotient by 1,000002072, the quotient is 1,000000219, which gives 9 in the 8th class for the 8th figure of the log.: and again the last quotient 1,000000219 being divided by 1,000000207, the next less, the quotient 1,000000012 gives 5 in the same 8th class, when one figure is cut off, for the 9th figure of the logarithm sought. All which figures collected together give 3,301029995 for Briggs's log. of 2000, the index 3 being supplied; which logarithm is true in the last figure.

To find the number answering to any given logarithm, as suppose to 3,3010300: omitting the characteristic, against the other figures 3, 0, 1, 0, 3, 0, 0, as in the first column in the margin, are the several numbers as in the 2d column, found from their respective 1st, 2d, 3d, &c classes; the effective numbers of which multiplied continually together, the last product is 2,000000019966, which, because the characteristic is 3, gives 2000,000019966, or 2000 only, for the required number, answering to the given logarithm.

3	1,995262315
0	0
1	1,002305238
0	0
3	1,000069080
0	0
0	0

Of Mr. Jones's Method.

In the 61st volume of the Philosophical Transactions, is a small paper on logarithms, which had been drawn up, and left unpublished, by the learned and ingenious William Jones, Esq. The method contained in this memoir, depends on an application of the doctrine of fluxions, to some properties drawn from the nature of the exponents of powers. Here all numbers are considered as some certain powers of a constant determinate root: so, any number x may be considered as the z power of any root r, or that $x = r^z$ is a general expression for all numbers, in terms of the constant root r, and a variable exponent z. Now the index z being the logarithm of the number x, therefore, to find this logarithm, is the same thing, as to find what power of the radical r is equal to the number x.

From this principle, the relation between the fluxions of any number x, and its logarithm z, is thus determined: Put $r = 1+n$; then is $x = r^z = (1+n)^z$, and $x + \dot{x} = (1+n)^{z+\dot{z}} = (1+n)^z \times (1+n)^{\dot{z}} = x \times (1+n)^{\dot{z}}$, which by expanding $(1+n)^{\dot{z}}$, omitting the 2d, 3d, &c powers of $\dot{z}$, and writing q for $\frac{n}{1+n}$, becomes $x + x\dot{z} \times (q + \frac{1}{2}q^2 + \frac{1}{3}q^3 + \frac{1}{4}q^4 + \&c)$; therefore $\dot{x} = ax\dot{z}$, putting a for the series $q + \frac{1}{2}q^2 + \frac{1}{3}q^3$ &c, or $f\dot{x} = x\dot{z}$, putting $f = \frac{1}{a}$.

Now when $r = 1+n = 10$, as in the common logarithms of Briggs's form; then $n = 9$, $q = ,9$, and the series $q + \frac{1}{2}q^2 + \frac{1}{3}q^3$ &c, gives $a = 2,302585$&c, and theref. its recip. $f = ,434294$&c. But if $a = 1 = f$, the form will be that of Napier's logarithms.

From the above form $x\dot{z}=f\dot{x}$, or $\dot{z} = \frac{f\dot{x}}{x}$, are then deduced many curious and general properties of logarithms, with the several series heretofore given by Gregory, Mercator, Wallis, Newton, and Halley. But of all these series, that one which our author selects for constructing the logarithms, is this, putting $N = \frac{r-p}{r+p}$, the logarithm of $\frac{r}{p}$ is $= 2f \times : N + \frac{1}{3}N^3 + \frac{1}{5}N^5 + \frac{1}{7}N^7 +$ &c, in the case in which $r - p$ is $= 1$, and consequently in that case $N = \frac{1}{2r-1}$ or $\frac{1}{2p+1}$; which series will then converge very fast.

Hence, having given any numbers, p, q, r, &c, and as many ratios a, b, c, &c, composed of them, the difference between the two terms of each ratio being 1; as also the logarithms A, B, C, &c, of those ratios given: to find the logarithms P, Q, R, &c, of those numbers; supposing $f = 1$. For instance, if $p = 2$, $q = 3$, $r = 5$; and $a = \frac{9}{8} = \frac{3^2}{2^3}$, $b = \frac{16}{15} = \frac{2^4}{3 \cdot 5}$, $c = \frac{25}{24} = \frac{5^2}{3 \cdot 2^3}$. Now the logarithms A, B, C, of these ratios a, b, c, being found by the above series, from the nature of powers we have these three equations,

$$\left.\begin{array}{l} A = 2Q - 3P \\ B = 4P - Q - R \\ C = 2R - Q - 3P \end{array}\right\} \text{which equations reduced give}$$

$$P = 3A + 4B + 2C = \text{log. of } 2.$$
$$Q = 5A + 6B + 3C = \text{log. of } 3.$$
$$R = 7A + 9B + 5C = \text{log. of } 5.$$

And hence $P + R = 10A + 13B + 7C$ is $=$ the logarithm of 2×5 or 10.

An elegant tract on logarithms, as a comment on Dr. Halley's method, was also given by Mr. Jones, in his Synopsis Palmariorum Matheseos, published in the year 1706. And, in the Philosophical Transactions, he communicated various improvements in goniometrical properties, and the series relating to the circle and to trigonometry.

The memoir above described was delivered to the Royal Society by their then librarian, Mr. John Robertson, a worthy, ingenious, and industrious man, who also communicated to the Society several little tracts of his own relating to logarithmical subjects; he was also the author of an excellent treatise on the Elements of Navigation in two volumes; and he was successively mathematical master to Christ's hospital in London; head master to the royal naval academy at Portsmouth; and librarian, clerk, and housekeeper, to the Royal Society; at whose house, in Crane Court, Fleet-street, he died in 1776, aged 64 years.

And among the papers of Mr. Robertson, I have, since his death, found one containing the following particulars relating to Mr. Jones, which I here insert, as I know of no other account of his life, &c, and as any true anecdotes of such extraordinary men must always be acceptable to the learned.—This paper is not in Mr. Robertson's hand writing, but in a kind of running law-hand, and is signed R. M. 12 Sept. 1771.

" William Jones, Esquire, F. R. S. was born at the foot of Bodavon mountain [Mynydd Bodafon], in the parish of Llanfihangel tre'r Bardd, in the isle of Anglesey, North Wales, in the year 1675. His father John George* was a farmer, of a good family, being descended from Hwfa ap Cynddelw, one of the 15 tribes of North Wales. He gave his two sons the common school education of the country, reading, writing, and accounts, in English, and the latin grammar. Harry his second soon took to the farming business; but William the eldest, having an extraordinary turn for mathematical studies, determined to try his fortune abroad from a place where the same was but of little service to him; he accordingly came to London, accompanied by a young man, Rowland Williams, afterwards an eminent perfumer in Wych-street. The report in the country is, that Mr. Jones soon got into a merchant's counting-house, and so gained the esteem of his master, that he gave him the command of a ship for a West-India voyage; and that upon his return he set up a mathematical school,

* " It is the custom in several parts of Wales for the name of the father to become the surname of his children. John George the father was commonly called Sion Siors of Llambado, to which parish he moved, and where his children were brought up."

and published his book of navigation*; and that upon the death of the merchant he married his widow: that Lord Macclesfield's son being his pupil, he was made secretary to the chancellor, and one of the D. tellers of the exchequer—and they have a story of an Italian wedding which caused great disturbance in Lord Macclesfield's family, but compromised by Mr. Jones; which gave rise to a saying, that Macclesfield was the making of Jones, and Jones the making of Macclesfield." Mr. Jones died July 3, 1749, being vice-president of the Royal Society; and left one daughter, and a young son, who was the late Sir William Jones, one of the judges in India, and highly esteemed for his great abilities, extensive learning, and eminent patriotism.

Of Mr. Andrew Reid and Others.

Andrew Reid, Esq. published in 1767 a quarto tract, under the title of An Essay on Logarithms, in which he also shows the computation of logarithms, from principles depending on the binomial theorem and the nature of the exponents of powers, the logarithms of numbers being here considered as the exponents of the powers of 10. He hence brings out the usual series for logarithms, and largely exemplifies Dr. Halley's most simple construction.

Besides the authors whose methods have been here particularly described, many others have treated on the subject of logarithms, and of the sines, tangents, secants, &c; among the principal of whom are Leibnitz, Euler, Maclaurin, Wolfius, and professor Simson, in an elegant geometrical tract on logarithms, contained in his posthumous works, printed in 4to at Glasgow, in the year 1776, at the expense of the very learned Earl Stanhope, and by his Lordship disposed of in

* This tract on navigation, intitled, " A New Compendium of the whole Art of Practical Navigation," was published in 1702, and dedicated " to the reverend and learned Mr. John Harris, M. A. and F. R. S." the author, I apprehend, of the " Universal Dictionary of Arts and Sciences," under whose roof Mr. Jones says he composed the said treatise on Navigation.

presents among gentlemen most eminent for mathematical learning.

Of Mr. Dodson's Anti-logarithmic Canon.

The only remaining considerable work of this kind published, that I know of, is the Anti-logarithmic Canon of Mr. James Dodson, an ingenious mathematician, which work he published in folio in the year 1742; a very great performance, containing all the logs. under 100000, and their corresponding natural numbers to 11 places of figures, with all their differences and the proportional parts; the whole arranged in the order contrary to that used in the common tables of numbers and logarithms, the exact logarithms being here placed first, and increasing continually by 1, from 1 to 100000, with their corresponding nearest numbers in the columns opposite to them; and, by means of the differences and proportional parts, the logarithm to any number, or the number to any logarithm, each to 11 places of figures, is readily found. This work contains also, besides the construction of the natural numbers to the given logarithms, "precepts and examples, showing some of the uses of logarithms, in facilitating the most difficult operations in common arithmetic, cases of interest, annuities, mensuration, &c; to which is prefixed an introduction, containing a short account of logarithms, and of the most considerable improvements made, since their invention, in the manner of constructing them."

The manner in which these numbers were constructed, consists chiefly in imitations of some of the methods before described by Briggs, and is nothing more than generating a scale of 100000 geometrical proportionals, from 1 the least term, to 10 the greatest, each continued to 11 places of figures; and the means of effecting this, are such as easily flow from the nature of a series of proportionals, and are briefly as follow. First, between 1 and 10 are interposed 9 mean proportionals; then between each of these 11 terms there are interposed 9 other means, making in all 101 terms; then between each of these a 3d set of 9 means, making in all 1001 terms; again between each of these a 4th set of 9 means, making in all 10001 terms; and lastly, between each two of these terms, a 5th set of 9 means, making in all 100001 terms, including both the 1 and the 10. The first four of these 5 sets of means, are found each by one extraction of the 10th root of the greater of the two given terms, which root is the least mean, and then multiplying it continually by itself, according to the number of terms in the section or set; and the 5th or last section is made by interposing each of the 9 means by help of the method of differences before taught. Namely, putting 10, the greatest term, $= A$, $A^{\frac{1}{10}} = B$, $B^{\frac{1}{10}} = C$, $C^{\frac{1}{10}} = D$, $D^{\frac{1}{10}} = E$, and $E^{\frac{1}{10}} = F$; now extracting the 10th root of A or 10, it gives $1{,}2589254118 = B = A^{\frac{1}{10}}$, for the least of the 1st set of means; and then multiplying it continually by itself, we have B, B^2, B^3, B^4, &c, to $B^{10} = A$, for all the 10 terms: 2dly, the 10th root of 1,2589254118 gives $1{,}0232929923 = C = B^{\frac{1}{10}} = A^{\frac{1}{100}}$, for the least of the 2d class of means; which being continually multiplied gives C, C^2, C^3, &c, to $C^{100} = B^{10} = A$, for all the 2d class of 100 terms: 3dly, the 10th root of 1,0232929923 gives $1{,}0023052381 = D = C^{\frac{1}{10}} = B^{\frac{1}{100}} = A^{\frac{1}{1000}}$, for the least of the 3d class of means; which being continually multiplied, gives D, D^2, D^3, &c, to $D^{1000} = C^{100} = B^{10} = A$, for the 3d class of 1000 terms: 4thly, the 10th root of 1,0023052381 gives $1{,}0002302850 = E = D^{\frac{1}{10}} = C^{\frac{1}{100}} = B^{\frac{1}{1000}} = A^{\frac{1}{10000}}$, for the least of the 4th class of means, which being continually multiplied, gives E, E^2, E^3, &c, to $E^{10000} = D^{1000} = C^{100} = B^{10} = A$, for the 4th class of 10000 terms. Now these 4 classes of terms, thus produced, require no less than 11110 multiplications of the least means by themselves; which however are much facilitated by making a small table of the first 10, or even 100 products, of the constant multiplier, and from it only taking out the proper lines, and adding them together: and these 4 classes of numbers always prove themselves at every 10th term, which must always agree with the corresponding successive terms of the preceding class. The remaining 5th class is constructed by means of differences, being much easier than the method of continual multiplication, the 1st and 2d differences only being used, as the 3d difference is too small to enter the computation of the sets of 9 means, between each two terms of the 4th class. And the several 2d differences, for each of these sets of 9 means, are found from the properties of a set of proportionals, $1, r, r^2, r^3$, &c, as disposed in the 1st column of the annexed table, and their several orders of differences as in the other columns of the table; where it is evident that

Terms.	1st dif.	2d dif.	3d dif.	&c.
$1\times$	$(r-1)\times$	$(r-1)^2\times$	$(r-1)^3\times$	
1	1	1	1	&c.
r	r	r	r	
r^2	r^2	r^2	r^2	
r^3	r^3	r^3	r^3	
&c.	&c.	&c.	&c.	

each column, both that of the given terms of the progression, and those of their orders of differences, forms a scale of proportionals, having the same common ratio r; and that each horizontal line, or row, forms a geometrical progression, having all the same common ratio $r-1$, which is also the 1st difference of each set of means: so, $(r-1)^2$ is the 1st of the 2d differences, and which is constantly the same, as the 3d differences become too small in the required terms of our progression to be regarded, at least near the beginning of the table: hence, like as 1, $r-1$, and $(r-1)^2$ are the 1st term, with its 1st and 2d differences; so r^n, $r^n.(r-1)$, and $r^n.(r-1)^2$, are any other term with its 1st and 2d differences. And by this rule the 1st and 2d differences are to be found, for every set of 9 means, viz, multiplying the 1st term of any class (which will be the several terms of the series E, E^2, E^3, &c, or every 10th term of the series F, F^2, F^3, &c) by $r-1$, or $F-1$, for the 1st difference, and this multiplied by $F-1$

again for the true 2d difference, at the beginning of that class. Thus, the 10th root of 1,0002302850, or E, gives 1,000023026116 for F, or the 1st mean of the lowest class, therefore $F - 1 = r - 1 =$,000023026116, is its 1st difference, and the square of it is $(r-1)^2 =$,0000000005302 its 2d diff.; then is ,000023026116F^{10n} or ,000023026116E^n, the 1st difference, and ,0000000005302F^{20n} or ,0000000005302E^{2n} is the 2d difference, at the beginning of the nth class of decades. And this 2d difference is used as the constant 2d difference through all the 10 terms, except towards the end of the table, where the differences increase fast enough to require a small correction of the 2d difference, which Mr. Dodson effects by taking a mean 2d difference among all the 2d differences, in this manner; having found the series of 1st differences $(F-1) . E^n$, $(F-1) . E^{n+1}$, $(F-1) . E^{n+2}$, &c, he takes the differences of these, and $\frac{1}{10}$ of them gives the mean 2d differences to be used, namely, $\frac{F-1}{10}(E^{n+1} - E^n)$, $\frac{F-1}{10}(E^{n+2} - E^{n+1})$, &c, are the mean 2d differences. And this is not only the more exact, but also the easier way. The common 2d difference, and the successive 1st differences, are then continually added, through the whole decade, to give the successive terms of the required progression.

TRACT XXII.

SOME PROPERTIES OF THE POWERS OF NUMBERS.

1. Of any two square numbers, at any distance from each other in the natural series of the squares 1^2, 2^2, 3^2, 4^2, &c, the mean proportional between the two squares, is equal to the less square plus its root multiplied by the difference of the roots, that is, by the distance in the series between the two square numbers, or by 1 more than the number of squares between them. The same mean proportional, is also equal to the greater of the two squares, minus its root the same