Machin found the circumference of a circle, whose diameter is 1, to be

3·14159265335,8979323846,2643383279,5028841971,6939937510,
5820974944,5923078164,0628620899,8628034825,3421170679 +,

true to above 100 places of figures.

Or, by substituting the above numbers in Machin's series, we get the series $(\frac{16}{5}-\frac{4}{239})-\frac{1}{3}(\frac{16}{5^3}-\frac{4}{239^3})+\frac{1}{5}(\frac{16}{5^5}-\frac{4}{239^5}$ &c, equal to the semicircumference whose radius is 1, or the whole circumference whose diameter is 1. Being the series published by Mr. Jones, and which he acknowledges he received from Mr. Machin.

But because the arc whose tangent is $\frac{1}{5}$, is $=$ 2 times the arc whose tangent is $\frac{1}{10}$, minus the arc to tangent $\frac{1}{515}$; (for $\frac{\frac{2}{10}}{1-\frac{1}{100}}=\frac{20}{99}=$ tangent of twice the arc to tangent $\frac{1}{10}$, and $\frac{\frac{20}{99}-\frac{1}{5}}{1+\frac{4}{99}}=\frac{1}{515}=$ tang. of diff. between the arcs whose tangents are $\frac{20}{99}$ and $\frac{1}{5}$); therefore 8 times arc to tangent $\frac{1}{10}-4$ times arc to tang. $\frac{1}{515}-$ arc to tang. $\frac{1}{239}=$ arc of 45°, or whose tang. is 1. Which is much easier than Machin's way. And various other methods may easily be discovered from the same principles.

TRACT XVIII.

A NEW AND GENERAL METHOD OF FINDING SIMPLE AND QUICKLY-CONVERGING SERIES; BY WHICH THE PROPORTION OF THE DIAMETER OF A CIRCLE TO ITS CIRCUMFERENCE MAY EASILY BE COMPUTED TO A GREAT MANY PLACES OF FIGURES.

In examining the methods of Mr. Machin and others, for computing the proportion of the diameter of a circle to its circumference, I discovered the method explained in this paper. This method is very general, and discovers many series that are fit for the abovementioned purpose. The advantage of this method is chiefly owing to the simplicity of the series by which an arc is found from its tangent. For, if t denote the tangent of an arc a, the radius being 1, then it is well known, that the arc a is denoted by the infinite series, $t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \frac{1}{9}t^9 - \&c$; where the form is as simple as can be desired. And it is evident that nothing further is required, than to contrive matters so, as that the value of the quantity t, in this series, may be both a small and a very simple number. Small, that the series may be made to converge sufficiently fast; and simple, that the several powers of t may be raised by easy multiplications, or easy divisions.

Since the first discovery of the above series, many authors have used it, and that after different methods, for determining the length of the circumference to a great number of figures. Among these were, Dr. Halley, Mr. Abra. Sharp, Mr. Machin, and others, of our own country; and M. de Lagney, M. Euler, &c, abroad. Dr. Halley used the arc of 30°, or $\frac{1}{12}$th of the circumference, the tangent of which being $=\sqrt{\frac{1}{3}}$, by substituting $\sqrt{\frac{1}{3}}$ for t in the above series, and multiplying by 6, the semicircumference is =

$$6\sqrt{\tfrac{1}{3}} \times (1 - \frac{1}{3.3} + \frac{1}{5.3^2} + \frac{1}{7.3^3} + \frac{1}{9.3^4} - \&c);$$ which series is, to be sure, very simple; but its rate of converging is not very great, on which account a great many terms must be used to compute the circumference to many places of figures. By this very series however, the industrious Mr. Sharp computed the circumference to 72 places of figures; Mr. Machin extended it to 100; and M. de Lagney, still by the same series, continued it to 128 places of figures. But though this series, from the 12th part of the circumference, does not converge very quickly, it is perhaps the best aliquot part of the circumference which can be employed for this purpose; for when smaller arcs, which are exact aliquot parts, are used, their tangents, though smaller, are so much more complex, as to render them, on the whole, more operose in the application: this will easily appear, by inspecting some instances

that have been given in the introductions to logarithmic tables. One of these methods is from the arc of 18°, the tangent of which is $\sqrt{(1-2\sqrt{\frac{1}{5}})}$; another is from the arc of $22°\frac{1}{2}$, the tangent of which is $\sqrt{2}-1$; and a third is from the arc of 15°, the tangent of which is $2-\sqrt{3}$. All of which are evidently too complex to afford an easy application to the general series.

In order to a still further improvement of the method by the above general series, Mr. Machin, by a very singular and excellent contrivance, has greatly reduced the labour naturally attending it. I have given an analysis of his method, or a conjecture concerning the manner in which it is probable Mr. Machin discovered it, in my Treatise on Mensuration; which, I believe, is the only book in which that method has been investigated, as it is repeated in the foregoing Tract. For though the series discovered by that method were published by Mr. Jones, in his "Synopsis Palmariorum Matheseos," which was printed in the year 1706, he has given them merely by themselves, without the least hint of the manner in which they were obtained. The result shows, that the proportion of the diameter to the circumference, is equal to that of 1 to quadruple the sum of the two series,

$$\frac{4}{5} \times \left(1 - \frac{1}{3.5^2} + \frac{1}{5.5^4} - \frac{1}{7.5^6} + \frac{1}{9.5^8} \text{ \&c}\right) \text{ and}$$

$$\frac{1}{239} \times \left(1 - \frac{1}{3.239^2} + \frac{1}{5.239^4} - \frac{1}{7.239^6} + \frac{1}{9.239^8} \text{ \&c}\right).$$

The slower of which series converges almost thrice as fast as Dr. Halley's, raised from the tangent of 30°. The latter of these two series converges still a great deal quicker; but then the large prime number 239, by the reciprocals of the powers of which the series converges, occasions such long and tedious divisions, as to counter-balance its quickness of convergency; so that the former series is summed with rather more ease than the latter, to the same number of places of figures. Mr. Jones, in his "Synopsis," mentions other series besides this, which he had received from Mr. Machin for the same purpose, and drawn from the same principle.

But we may conclude this to be the best of them all, as he did not publish any other besides it.

M. Euler too, in his "Introductio in Analysin Infinitorum," by a contrivance something like Mr. Machin's, discovers, that $\frac{1}{2}$ and $\frac{1}{3}$ are the tangents of two arcs, the sum of which is just 45°; and that therefore the diameter is to the circumference, as 1 to quadruple the sum of the two following series,

$$\frac{1}{2} \times (1 - \frac{1}{3.4} + \frac{1}{5.4^2} - \frac{1}{7.4^3} + \frac{1}{9.4^4} \text{ \&c) and}$$

$$\frac{1}{3} \times (1 - \frac{1}{3.9} + \frac{1}{5.9^2} - \frac{1}{7.9^3} + \frac{1}{9.9^4} \text{ \&c}).$$

Both which series converge much faster than Dr. Halley's, and are yet at the same time made to converge by the powers of numbers producing only short divisions; that is, divisions performed in one line, or without writing down any thing besides the quotients.

I come now to explain my own method, which indeed bears some little resemblance to the methods of Machin and Euler; but then it is more general, and discovers, as particular cases of it, both the series of those gentlemen, and many others, some of which are fitter for this purpose than theirs are.

This method then consists in finding out such small arcs, as have for tangents some small and simple vulgar fractions, the radius being denoted by 1, and such also that some multiple of those arcs shall differ from an arc of 45°, the tangent of which is equal to the radius, by other small arcs, which also shall have tangents denoted by other such small and simple vulgar fractions. For it is evident, that if such a small arc can be found, some multiple of which has such a proposed difference, from an arc of 45°, then the lengths of these two small arcs will be easily computed from the general series, because of the smallness and simplicity of their tangents; after which, if the proper multiple of the first arc be increased or diminished by the other arc, the result will be the length of an arc of 45°, or $\frac{1}{8}$th of the circumference. And the manner in which I discover such arcs is thus:

Let T, t, denote any two tangents, of which T is the greater, and t the less: then it is known, that the tangent of the difference of the corresponding arcs is equal to $\frac{T-t}{1+Tt}$. Hence, if t, the tangent of the smaller arc, be successively denoted by each of the simple fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c, the general expression for the tangent of the difference between the arcs will become respectively $\frac{2T-1}{2+T}$, $\frac{3T-1}{3+T}$, $\frac{4T-1}{4+T}$, $\frac{5T-1}{5+T}$, &c; so that if T be expounded by any given number, then these expressions will give the tangent of the difference of the arcs in known numbers, according to the values of t, severally assumed respectively. And if, in the first place, T be equal to 1, the tangent of 45°, the foregoing expressions will give the tangent of an arc, which is equal to the difference between that of 45° and the first arc; or that of which the tangent is one of the numbers $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c. Then, if the tangent of this difference, just now found, be taken for T, the same expressions will give the tangent of an arc, which is equal to the difference between the arc of 45° and the double of the first arc. Again, if for T we take the tangent of this last found difference, then the foregoing expressions will give the tangent of an arc, equal to the difference between that of 45° and the triple of the first arc. And again taking this last found tangent for T, the same theorem will produce the tangent of an arc equal to the difference between that of 45° and the quadruple of the first arc; and so on, always taking for T the tangent last found, the same expressions will give the tangent of the difference between the arc of 45° and the next greater multiple of the first arc; or that of which the tangent was at first assumed equal to one of the small numbers $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c. This operation, being continued till some of the expressions give such a fit, small, and simple fraction as is required, is then at an end, for we have then found two such small tangents as were required, viz, the tangent last found, and the tangent first assumed.

Here follow the several operations adapted to the several values of t. The letters a, b, c, d, &c, denote the several successive tangents.

1. Take $t = \frac{1}{2}$, then the theorem $\frac{2T-1}{2+T}$ gives $a = \frac{1}{3}$, $b = -\frac{1}{7}$. Therefore the arc of 45°, or $\frac{1}{8}$th of the circumference, is either equal to the sum of the two arcs of which $\frac{1}{2}$ and $\frac{1}{3}$ are the tangents, or to the difference between the arc of which the tangent is $\frac{1}{7}$, and the double of the arc of which the tangent is $\frac{1}{2}$; that is, putting A = the arc of 45°, then

$$A = \begin{cases} \frac{1}{2} \times (1 - \frac{1}{3.4} + \frac{1}{5.4^2} - \frac{1}{7.4^3} + \frac{1}{9.4^4} - \text{\&c.}) \\ + \frac{1}{3} \times (1 - \frac{1}{3.9} + \frac{1}{5.9^2} - \frac{1}{7.9^3} + \frac{1}{9.9^4} - \text{\&c.}) \end{cases}$$

$$\text{Or, } A = \begin{cases} 1 - \frac{1}{3.4} + \frac{1}{5.4^2} - \frac{1}{7.4^3} + \frac{1}{9.4^4} - \text{\&c,} \\ - \frac{1}{7} \times (1 - \frac{1}{3.49} + \frac{1}{5.49^2} - \frac{1}{7.49^3} + \frac{1}{9.49^4} - \text{\&c).} \end{cases}$$

The former of these values of A is the same with that before mentioned, as given by M. Euler; but the latter is much better, as the powers of $\frac{1}{49}$ converge much faster than those of $\frac{1}{9}$.

Corol.—From double the former of these values of A, subtracting the latter, the remainder is,

$$A = \begin{cases} \frac{2}{3} \times (1 - \frac{1}{3.9} + \frac{1}{5.9^2} - \frac{1}{7.9^3} + \text{\&c.}) \\ + \frac{1}{7} \times (1 - \frac{1}{3.49} + \frac{1}{5.49^2} - \frac{1}{7.49^3} + \text{\&c.}) \end{cases}$$

which is a much better theorem than either of the former.

2. If t be taken $= \frac{1}{3}$, then the expression $\frac{3T-1}{3+T}$ gives $a = \frac{1}{2}$, $b = \frac{1}{7}$. Here the value of $a = \frac{1}{2}$ gives the same expression for the value of A as the first in the foregoing case, and the value of $b = \frac{1}{7}$ gives the value of A the very same as in the corollary to the case above.

3. Taking $t = \frac{1}{4}$, the expression $\frac{4T-1}{4+T}$ gives $a = \frac{3}{5}$, $b = \frac{7}{23}$, $c = \frac{5}{99}$, $d = -\frac{79}{401}$. Where it is evident that the value

of $c = \frac{5}{99}$ is the fittest number afforded by this case; and hence it appears, that the arc of 45° is equal to the sum of the arc of which the tangent is $\frac{5}{99}$, and the triple of the arc of which the tangent is $\frac{1}{4}$.

$$\text{Or that A} = \begin{cases} \frac{3}{4} \times (1 - \frac{1}{3.16} + \frac{1}{5.16^2} - \frac{1}{7.16^3} + \&c) \\ + \frac{5}{99} \times (1 - \frac{5^2}{3.99^2} + \frac{5^4}{5.99^4} - \frac{5^6}{7.99^6} + \&c). \end{cases}$$

Which is the best theorem that we have yet found, because the number 99 resolves into the two easy factors 9 and 11.

4. Let now t be taken $= \frac{1}{5}$; then the expression $\frac{5T-1}{5+T}$ gives $a = \frac{2}{3}$, $b = \frac{7}{17}$, $c = \frac{9}{46}$, $d = -\frac{1}{239}$. Where it is evident that the last number, or the value of d, is the fittest of those produced in this case; and from which it appears, that the arc of 45° is equal to the difference between the arc of which the tangent is $\frac{1}{239}$, and quadruple the arc of which the tangent is $\frac{1}{5}$. Or that

$$\text{A} = \begin{cases} \frac{4}{5} \times (1 - \frac{1}{3.5^2} + \frac{1}{5.5^4} - \frac{1}{7.5^6} + \&c.) \\ - \frac{1}{239} \times (1 - \frac{1}{3.239^2} + \frac{1}{5.239^4} - \frac{1}{7.239^6} + \&c). \end{cases}$$

Which is the very theorem that was invented by Mr. Machin, as we have before mentioned.

5. Take now $t = \frac{1}{6}$; then the expression $\frac{6T-1}{6+T}$ gives $a = \frac{5}{7}$, $b = \frac{23}{47}$, $c = \frac{91}{305}$, $d = \frac{241}{1921}$, $e = \frac{-475}{11767}$. Of which numbers it is evident that none are fit for our purpose.

6. Again, take $t = \frac{1}{7}$, and the expression $\frac{7T-1}{7+T}$ will give $a = \frac{3}{4}$, $b = \frac{17}{31}$, $c = \frac{11}{28}$, $d = \frac{49}{205}$, $e = \frac{69}{742}$, $f = -\frac{259}{5263}$. Neither are any of these numbers fit for our purpose.

7. In like manner take $t = \frac{1}{8}$, so shall $\frac{8T-1}{8+T}$ give $a = \frac{7}{9}$, $b = \frac{47}{79}$, $c = \frac{297}{679}$, $d = \frac{1697}{5729}$, $e = \frac{7847}{47529}$, $f = \frac{14047}{388079}$.

8. And if t be taken $= \frac{1}{9}$, the expression $\frac{9T-1}{9+T}$ will give
$a = \frac{4}{5}$, $b = \frac{31}{49}$, $c = \frac{115}{236}$, $d = \frac{799}{2239}$, $e = \frac{2467}{10475}$, &c.

9. Also, if we take $t = \frac{1}{10}$, the expression $\frac{10T-1}{10+T}$ will give
$a = \frac{9}{11}$, $b = \frac{79}{119}$, $c = \frac{671}{1269}$, $d = \frac{5441}{13361}$, $e = \frac{41049}{139051}$, &c.

10. Further, if we take $t = \frac{1}{11}$, the expression $\frac{11T-1}{11+T}$ gives
$a = \frac{5}{6}$, $b = \frac{49}{71}$, $c = \frac{234}{415}$, $d = \frac{2159}{4799}$, $e = \frac{9475}{27474}$, &c.

11. Lastly, if we take $t = \frac{1}{12}$, the expression $\frac{12T-1}{12+T}$ gives
$a = \frac{11}{13}$, $b = \frac{113}{167}$, $c = \frac{41}{73}$, $d = \frac{419}{917}$, $e = \frac{4111}{11423}$, &c.

Here it is evident, that none of these latter cases afford any numbers that are fit for this purpose. And to try any other fractions less than $\frac{1}{12}$ for the value of t, does not seem likely to answer any good purpose, especially as the divisors after 12 become too large to be managed in the easy way of short division in one line.

By the foregoing means it appears then, that we have discovered five different forms of the value of A, or $\frac{1}{4}$th of the semicircumference, all of which are very proper for readily computing its length; viz, three forms in the first case and its corollary, one in the 3d case, and one in the 4th case. Of these, the first and last are the same as those invented by Euler and Machin respectively, and the other three are quite new, as far as I know.

But another remarkable excellence attending the first three of the before mentioned series, is, that they are capable of being changed into others which not only converge still faster, but in which the converging quantity shall be $\frac{1}{10}$, or some multiple or sub-multiple of it, and so the powers of it raised with the utmost ease. The series, or theorems, here meant are these three:

$$1st,\ A = \begin{cases} \frac{1}{2} \times (1 - \frac{1}{3.4} + \frac{1}{5.4^2} - \frac{1}{7.4^3} + \&c) \\ + \frac{1}{3} \times (1 - \frac{1}{3.9} + \frac{1}{5.9^2} - \frac{1}{7.9^3} + \&c). \end{cases}$$

$$2dly,\ A = \begin{cases} 1 - \frac{1}{3.4} + \frac{1}{5.4^2} - \frac{1}{7.4^3} + \&c \\ - \frac{1}{7} \times (1 - \frac{1}{3.49} + \frac{1}{5.49^2} - \frac{1}{7.49^3} + \&c). \end{cases}$$

$$3dly,\ A = \begin{cases} \frac{2}{3} \times (1 - \frac{1}{3.9} + \frac{1}{5.9^2} - \frac{1}{7.9^3} + \&c) \\ + \frac{1}{7} \times (1 - \frac{1}{3.49} + \frac{1}{5.49^2} - \frac{1}{7.49^3} + \&c). \end{cases}$$

Now if each of these be transformed, by means of the differential series, in cor. 3 p. 64 of the late Mr. Simpson's Mathematical Dissertations, they will become of these very commodious forms, viz,

$$1st,\ A = \begin{cases} \frac{4}{10} \times (1 + \frac{4}{3.10} + \frac{8\alpha}{5.10} + \frac{12\beta}{7.10} + \&c) \\ + \frac{3}{10} \times (1 + \frac{2}{3.10} + \frac{4\alpha}{5.10} + \frac{6\beta}{7.10} + \&c). \end{cases}$$

$$2dly,\ A = \begin{cases} \frac{4}{5} \times (1 + \frac{4}{3.10} + \frac{8\alpha}{5.10} + \frac{12\beta}{7.10} + \&c) \\ - \frac{7}{50} \times (1 + \frac{4}{3.100} + \frac{8\alpha}{5.100} + \frac{12\beta}{7.100} + \&c). \end{cases}$$

$$3dly,\ A = \begin{cases} \frac{6}{10} \times (1 + \frac{2}{3.10} + \frac{4\alpha}{5.10} + \frac{6\beta}{7.10} + \&c) \\ + \frac{7}{50} \times (1 + \frac{2}{3.50} + \frac{4\alpha}{5.50} + \frac{6\beta}{7.50} + \&c). \end{cases}$$

Where α, β, γ, &c, denote always the preceding terms in each series.

Now it is evident that all these latter series are much easier than the former ones, to which they respectively correspond; for, because of the powers of 10 here concerned, we have little more to do than to divide by the series of odd numbers 1, 3, 5, 7, 9, &c.

Of all these three forms, the 2d is the fittest for computing the required proportion; because, of the two series of which it consists, the several terms of the one are found from the like terms of the other, by dividing these latter by 10, and its several successive powers, 100, 1000, &c; that is, the terms of the one consist of the same figures as the terms of the other, only removed a certain number of places farther towards the right hand, in the decuple scale of numbers; and the number of places by which they must be removed, is the same as the distance of each term from the first term of the series, viz, in the 2d term the figures must be moved one place lower, in the 3d term two, in the 4th term three, &c; so that the latter series will consist of but about half the number of the terms of the former. Thus then this method may be said to effect the business by one series only, in which there is little more to do, than to divide by the several numbers 1, 3, 5, 7, &c; for as to the multiplications by the numbers in the numerators of the terms, after they become large, they are easily performed by barely multiplying by the number 2, and subtracting one number from another: for since every numerator is less by 2 than the double of its denominator, if d denote any denominator, exclusive always of the powers of 10, then the coefficient of that term is $\frac{2d-2}{d}$ or $2 - \frac{2}{d}$, by which the preceding term is to be multiplied; to do which therefore, multiply it by 2, that is double it, and divide that double by the divisor d, and subtract the quotient from the said double.