of parts, and through the 2d point of division draw EFG, so will AG be one of the equal parts very nearly.

Corol. 3.—The number 1·752 being equal to $\sqrt{3}$ nearly, for $\sqrt{3} = 1·732$; therefore, if DE be taken to DA as $\sqrt{3}$ to 1, the point E will be found answering the same purpose as before, but not quite so near as the former. And here, because DA : DE :: 1 : $\sqrt{3}$, therefore DE is the perpendicular of an equilateral triangle described on AC. Hence then, if with the centres A, C, and radius AC, two arcs be described, they will intersect in the point E, nearly the same as before. And this is the method in common practice; but it is not so near the truth as the construction in the 2d Corollary.

Corol. 4.—Hence also a right line is found equal to the arc of a circle nearly: for BG is $= \frac{11}{7}$ DF nearly. And this is the same as the ratio of 11 to 7, which Archimedes gave for the ratio of the semicircumference to the diameter, or 22 to 7 the ratio of the whole circumference to the diameter. But the proportion is here rendered general for any arc of the circle, as well as for the whole circumference.

TRACT XVI.

ON PLANE TRIGONOMETRY WITHOUT TABLES.

THE cases of trigonometry are usually calculated by means of tables of sines, tangents or secants, either of their natural numbers, or their logarithms. But the calculations may also be made without any such tables, to a tolerable degree of accuracy, by means of the theorems and rules contained in the following propositions and corollaries.

PROPOSITION.

If $2a$ denote a side of any triangle, A the number of degrees contained in its opposite angle, and r the radius of the circle circumscribing the triangle: Then the value of A is equal to

$$57{\cdot}2957795 \times : \frac{a}{r} + \frac{a^3}{2.3r^3} + \frac{3a^5}{2.4.5r^5} + \frac{3.5a^7}{2.4.6.7r^7} + \frac{3.5.7a^9}{2.4.6.8.9r^9} \ \&c.$$

For, since $2a$ is the chord of the arc on which the angle, whose measure is A, insists; a will be the sine of half that arc, or the sine of the angle to the radius r, since an angle in the circumference of a circle is measured by half the arc on which it stands; now it is well known that the said half arc z is equal to

$a + \frac{a^3}{2.3r^2} + \frac{3a^5}{2.4.5r^4} + \frac{3.5a^7}{2.4.6.7r^6}$ &c; and, $3{\cdot}14159r$ denoting half the circumference of the same circle, or the arc of 180 degrees, it will be

$$\text{as } 3{\cdot}14159r \ : \ 180^\circ \ :: \ z \ : \ \frac{180z}{3{\cdot}14159r} = \frac{57{\cdot}2957795z}{r}$$

$$= 57{\cdot}2957795 \times \left(\frac{a}{r} + \frac{a^3}{2.3r^3} + \frac{3a^5}{2.4.5r^5} + \frac{3.5a^7}{2.4.6.7r^7} \ \&c,\right)$$ the degrees in the angle or half arc.

Corollary 1.—By reverting the above series, we obtain

$$\frac{a}{r} = \frac{A}{n} - \frac{A^3}{2.3n^3} + \frac{A^5}{2.3.4.5n^5} - \frac{A^7}{2.3.4.5.6.7n^7} \ \&c;$$

$$\text{putting } n = 57{\cdot}2957795 = \frac{180}{3{\cdot}14159 \ \&c.}$$

Corollary 2.—If $2a$ be the hypothenuse of a right-angled triangle, a will be $= r$, and then the general series will become $n \times (a +$

$$\frac{1}{2.3} + \frac{3}{2.4.5} + \frac{3.5}{2.4.6.7} \ \&c) = 90, \text{ or } \frac{90}{n} = \frac{90 \times 3{\cdot}14159 \ \&c}{180} =$$

$$\frac{3.14159 \ \&c}{2} = 1 + \frac{1}{2.3} + \frac{3}{2.4.5} + \frac{3.5}{2.4.6.7} + \frac{2.5\ 7}{2.4.6.8.9} \ \&c.$$

Corol. 3.—Since the chord of 60 degrees is = the radius, or the sine of 30 degrees = half the radius, putting a for $\frac{1}{2}r$ in the general series, will give $n \times \left(\frac{1}{2} + \frac{1}{2.3.2^3} + \frac{3}{2.4.5.2^5} + \frac{3.5}{2.4.6.7.2^7}\right.$ &c $= 30$; and hence the sum of the infinite series

$$\frac{1}{2} + \frac{1}{2.3.2^3} + \frac{3}{2.4.5.2^5} + \frac{3.5}{2.4.6.7.2^7} \text{ \&c,}$$

$$\text{is} = \frac{30}{n} = \frac{30 \times 3{\cdot}14159 \text{ \&c}}{180} = \frac{3{\cdot}14159 \text{ \&c}}{6} =$$

$\frac{1}{6}$th of the circumference of the circle whose diameter is 1.

Corol. 4.—It might easily be shown, from the principles of common geometry, that the sine of 60 degrees is to the radius, as $\frac{1}{2}\sqrt{3}$ is to 1; substituting then $\frac{1}{2}r\sqrt{3}$ for a in the general series, we shall have $n\sqrt{3} \times (\frac{1}{2} + \frac{3}{2.3.2^3} + \frac{3.3^2}{2.4.5.2^5} + \frac{3.5.3^3}{2.4.6.7.2^7}$ &c) $= 60$; and hence the sum of the infinite series

$$\frac{1}{2} + \frac{3}{2.3.2^3} + \frac{3.3^2}{2.4.5.2^5} + \frac{3.5.3^3}{2.4.6.7.2^7} \text{ \&c, will be}$$

$= \frac{60}{n\sqrt{3}} = \frac{60 \times 3{\cdot}14159 \text{ \&c}}{180\sqrt{3}} = \frac{3{\cdot}14159 \text{ \&c}}{3\sqrt{3}}$, and is therefore to the infinite series in the 3d corollary, as 2 is to $\sqrt{3}$.

Corol. 5.—If b, c be the halves of the other two sides of the triangle, and B, C the degrees contained in their opposite angles; since $\text{B} = n \times (\frac{b}{r} + \frac{b^3}{2.3r^3} + \frac{3b^5}{2.4.5r^5}$ &c), and $\text{C} = n \times (\frac{c}{r} + \frac{c^3}{2.3r^3}$ &c, and the 3 angles of any triangle are equal to 180 degrees; we shall have $180 = \text{A} + \text{B} + \text{C} = n \times (\frac{a+b+c}{r} + \frac{a^3+b^3+c^3}{2.3r^3}$ &c), or the sum of the infinite series

$$\frac{a+b \text{ c}}{r} + \frac{1}{2.3} \cdot \frac{a^3+b^3+c^3}{r^3} + \frac{3}{2.4.5} \cdot \frac{a^5+b^5+c^5}{r^5} + \frac{3.5}{2.4.6.7} \cdot \frac{a^7+b^7+c^7}{r^7}$$

&c, will be $= \frac{180}{n} = \frac{180 \times 3{\cdot}14159 \text{ \&c}}{180} = 3.14159$ &c $=$ the circumference of a circle whose diameter is 1; a, b, c, being the halves of the three sides of any triangle, and r the radius of its circumscribing circle.

Corol. 6.—Since, by theor. 3, $b : a + c :: a - c : \frac{aa - cc}{b} =$ half the difference of the segments of the base (b) made by a perpendicular demitted from its opposite angle, and $b + \frac{aa-cc}{b} = \frac{aa+bb-cc}{b}$ = the segment adjoining to the side $2a$, we shall have $\sqrt{(4a^2 - \frac{(aa+bb-cc)^2}{bb})} = \frac{\sqrt{(4a^2b^2-(aa+bb-cc)^2)}}{b}$ for the value of the said perpendicular to the base; and hence $\frac{\sqrt{(4a^2b^2-(aa+bb-cc)^2)}}{b} : 2a :: c : \frac{2abc}{\sqrt{(4a^2b^2-(aa+bb-cc)^2)}} = r$ the radius of the circumscribing circle.

Having now found the value of r, we can calculate all the cases of trigonometry without any tables, and without reducing oblique triangles to right-angled ones; for, having any three parts (except the three angles) given, we can find the rest from these five equations following:

1. $r = \frac{2abc}{\sqrt{(4a^2b^2-(aa+bb-cc)^2)}}$.

2. $\text{A} = n \times (\frac{a}{r} + \frac{a^3}{2.3r^3} + \frac{3a^5}{2.4.5r^5} + \frac{3.5a^7}{2.4.6.7r^7} + \frac{3.5.7a^9}{2.4.6.8.9r^9}$ &c.)

3. $\text{B} = n \times (\frac{b}{r} + \frac{b^3}{2.3r^3} + \frac{3b^5}{2.4.5r^5} + \frac{3.5b^7}{2.4.6.7r^7} + \frac{3.5.7b^9}{2.4.6.8.9r^9}$ &c.)

4. $\text{C} = n \times (\frac{c}{r} + \frac{c^3}{2.3r^3} + \frac{3c^5}{2.4.5r^5} + \frac{3.5c^7}{2.4.6.7r^7} + \frac{3.5.7c^9}{2.4.6.8.9r^9}$ &c.)

5. $\text{A} + \text{B} + \text{C} = 180$.

And, for the more convenience, we may add the three following, which are derived from the 2d, 3d, and 4th, by reversion of series.

6. $a = r \times (\frac{\text{A}}{n} - \frac{\text{A}^3}{2.3n^3} + \frac{\text{A}^5}{2.3.4.5n^5} - \frac{\text{A}^7}{2.3.4.5.6.7n^7}$ &c.)

7. $b = r \times (\frac{\text{B}}{n} - \frac{\text{B}^3}{2.3n^3} + \frac{\text{B}^5}{2.3.4.5n^5} - \frac{\text{B}^7}{2.3.4.5.6.7n^7}$ &c.)

8. $c = r \times (\frac{\text{C}}{n} - \frac{\text{C}^3}{2.3n^3} + \frac{\text{C}^5}{2.3.4.5n^5} - \frac{\text{C}^7}{2.3.4.5.6.7n^7}$ &c.)

Where $n = 57{\cdot}2957795$ &c.

EXAMPLE.

Suppose we take here the following example, in which are given the two sides $2b = 345$, $2c = 232$, and the angle opposite to $2c = 37° 20' = 37\frac{1}{3}$ degrees = C. Then since $\frac{C}{n} = \frac{37\frac{1}{3} \times 3{\cdot}14159 \text{ \&c}}{180} = {\cdot}651589587$, we have $c = \frac{232}{2}$ $= 116 = r \times ({\cdot}651589587 - {\cdot}04610744 + {\cdot}00097879 - {\cdot}000009894 + {\cdot}000000058 \text{ \&c}) = r \times ({\cdot}652568435 - {\cdot}046117334) = {\cdot}6064511r$. Hence $r = \frac{116}{{\cdot}6064511} = 191{\cdot}27677$; and $\frac{b}{r} = \frac{345 \times {\cdot}6064511}{2 \times 116} = {\cdot}9018346$.

Again, $B = 57{\cdot}2957795 \times 1{\cdot}12402$ (the sum of the series in the 3d equation) $= 64{\cdot}4016$ degrees $= 64° 24'$.

And $A = 180 - 37\frac{1}{3} - 64{\cdot}4016 = 180 - 101{\cdot}735 = 78°{\cdot}265$ $= 78° 16'$ nearly.

Lastly, $\frac{A}{n}$ being $= \frac{78{\cdot}265}{57{\cdot}2957795} = 1{\cdot}365982$, and $r =$ $191{\cdot}27677$, from the 5th equation we have $a = 191{\cdot}27677 \times$ $(1{\cdot}365982 - {\cdot}4247992 + {\cdot}0396379 - {\cdot}0017607 + {\cdot}0000288 - {\cdot}0000005) = 191{\cdot}27677 \times {\cdot}9790883 = 187{\cdot}27684$.
And hence $2a = 374{\cdot}55368 =$ the third side of the triangle.

Corol. 7.—As the series by which an angle is found, often converges very slowly, I have inserted the following approximation of it; viz.

$A = n \times (\frac{4}{3}\sqrt{(2 - 2\sqrt{(1 - \frac{aa}{rr})})} - \frac{a}{3r})$ nearly; where the letters denote the same quantities as in the above series. For since $P = \sqrt{(2 - 2\sqrt{(1 - \frac{aa}{rr})})}$ is $= \frac{a}{r} + \frac{a^3}{2.4r^3} + \frac{7a^5}{2.4.16r^5}$ &c,

and $\frac{A}{n}$ is $= \frac{a}{r} + \frac{a^3}{2.3r^3} + \frac{3a^5}{2.4.5^5}$ &c,

we shall have, by taking the former of these from the latter,
$\frac{A}{n} - P = \frac{a^3}{24r^3} + \frac{13a^5}{640r^5}$ &c. But, from the first series,
$\frac{1}{3}P - \frac{a}{3r} = \frac{a^3}{24r^3} + \frac{7a^5}{384r^5}$ &c; hence, by subtracting the latter from the former, it gives

$\frac{\text{A}}{n} - \text{P} - \frac{1}{3}\text{P} + \frac{a}{3r} = \frac{a}{n} - \frac{4}{3}\text{P} + \frac{a}{3r} = \frac{a^5}{480r^5}$ &c ; and

$\text{A} = n \times (\frac{4}{3}\text{P} - \frac{a}{3r} = n \times (\frac{4}{3}\sqrt{(2-2\sqrt{(1-\frac{aa}{rr})})} - \frac{a}{3r})$ nearly.

Corol. 7.—And again, since $\frac{4}{105} \times (\text{P} - q - \frac{1}{8}q^3 = \frac{1}{480}q^5$ &c ; where q is $= \frac{a}{r}$; by subtracting this from $\frac{\text{A}}{n} - \frac{4\text{P}-q}{3} = \frac{1}{480}q^6$ &c, and reducing, there will be obtained $\text{A} = \frac{n}{105} \times (144\text{P} - 39q - \frac{1}{2}q^3) = \frac{n}{105} \times (144\sqrt{2-2\sqrt{(1-q^2)}}) - 39q - \frac{1}{2}q^5$, which will commonly give the angle exact to within a minute of the truth. Where note, that the constant quantity $\frac{n}{105}$ is $= \cdot 54567409$. And from the whole may be drawn the following general problem.

PROBLEM.

To perform all the Cases of Trigonometry without any Tables.

Having any three parts of a triangle given, except the three angles, the other three parts may be found, by some of the following six general theorems.

1. $\text{A} = \frac{1}{3}n \times (4\sqrt{(2-2\sqrt{(-\frac{a^2}{r^2})})} - \frac{a}{r})$ nearly. Or

$\text{A} = \frac{n}{105} \times (144\sqrt{(2-2\sqrt{(1-\frac{a^2}{r^2})})} - 39\frac{a}{r} - \frac{a^3}{2r^3})$ more nearly.

2. $\text{A} = n \times (\frac{a}{r} + \frac{a^3}{2.3r^3} + \frac{3a^5}{2.4.5r^5} + \frac{3.5a^7}{2.4.6.7r^7} + \frac{3.5.7a^9}{2.4.6.8.9r^9}$ &c.)

3. $a = r \times (\frac{\text{A}}{n} - \frac{\text{A}^3}{2.3n^3} + \frac{\text{A}^5}{2.3.4.5n^5} - \frac{\text{A}^7}{2.3.4.5.6.7n^7}$ &c.)

4. $r = \cfrac{a}{\frac{\text{A}}{n} - \frac{\text{A}^3}{2.3.n^3} + \frac{\text{A}^5}{2.3.4.5n^5} - \frac{\text{A}^7}{2.3.4.5.6.7n^7} \text{ \&c.}}$

5. $r = \frac{2abc}{\sqrt{(a^2b^2 - (a^2 + b^2 - c^2)^2)}}$

$= \frac{2abc}{\sqrt{[(a+b+c)\times(a+b-c)\times(a-b+c)\times(-a+b+c)]}}$.

6. $c = \sqrt{[a^2 + b^2 - 2ab\sqrt{(1 - (\frac{C}{n} - \frac{C^3}{2.3n^3} - \frac{C^5}{2.3.4.5n^5} \&c)^2]}}$.

Where a, b, c, are the halves of the three sides of the triangle, and A the number of degrees in the angle opposite the side $2a$, and C the degrees in the angle opposite the side $2c$; also r is the radius of the circumscribed circle;

and $n = \frac{180}{3{\cdot}14159} = 57{\cdot}2957795$, or $\frac{n}{105} = {\cdot}54567409$.

EXAMPLE.

Thus, if the three sides be given, as for example $a = 13$, $b = 14$, $c = 15$. Then is $r = 16\frac{1}{4}$, and the angles by these theorems come out as follow; viz.

Angles by the Theor.			The true Angles.	
53°	7′	- - angle A - -	53°	$7'\frac{4}{5}$
59	28	- - angle B - -	59	$29\frac{2}{5}$
67	19	- - angle C - -	67	$22\frac{4}{5}$
179	54	sum of all	180	00

TRACT XVII,

ON MACHIN'S QUADRATURE OF THE CIRCLE.

SINCE the chief advantage of this method consists in taking small arcs, whose tangents shall be numbers easy to manage, Mr. Machin very properly considered, that as the tangent of 45° is 1; and that the tangent of any arc being given, the tangent of double that arc can easily be found; if there be assumed some small simple number for the tangent of an arc,