TRACT XIV.

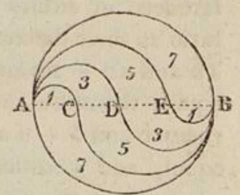
ON THE GEOMETRICAL DIVISION OF CIRCLES AND ELLIPSES INTO ANY NUMBER OF PARTS, AND IN ANY PROPOSED RATIOS.

ARTICLE 1.

In the year 1774 was published a pamphlet in 8vo, with this title, *A Dissertation on the Geometrical Analysis of the Antients. With a Collection of Theorems and Problems, without Solutions, for the Exercise of Young Students.* This pamphlet was anonymous; it was however well known to myself, and to several other persons, that the author of it was the late Mr. John Lawson, B. D. rector of Swanscombe in Kent, an ingenious and learned geometrician, and, what is still more estimable, a most worthy and good man; one in whose heart was found no guile, and whose pure integrity, joined to the most amiable simplicity of manners, and sweetness of temper, gained him the affection and respect of all who had the happiness to be acquainted with him. His collection of problems in that pamphlet concluded with this singular one, "To divide a circle into any number of parts, which shall be as well equal in area as in circumference.— N. B. *This may seem a paradox, however it may be effected in a manner strictly geometrical.*" The solution of this seeming paradox he reserved to himself, as far as I know; but I fell upon the discovery of it soon after; and my solution was published in an account which I gave of the pamphlet in the Critical Review for 1775, vol. xl, and which the author afterwards informed me was on the same principle as his own. This account is in page 21 of that volume, and in the following words:

2. "We have no doubt but that our mathematical readers will agree with us in allowing the truth of the author's remark concerning the seeming paradox of this problem; because there is no geometrical method of dividing the circumference of a circle into any proposed number of parts taken at pleasure, and it does not readily appear that there can be any other way of resolving the problem, than by drawing radii to the points of equal division in the circumference. However another method there is, and that strictly geometrical, which is as follows.

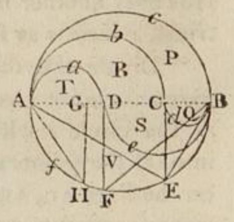
"Divide the diameter AB of the given circle into as many equal parts as the circle itself is to be divided into, at the points C, D, E, &c. Then on the lines AC, AD, AE, &c, as diameters, as also on BE, BD, BC, &c, describe semicircles, as in the annexed figure: and they will divide the whole circle in the manner as required.



"For, the several diameters being in arithmetical progression, of which the common difference is equal to the least of them, and the diameters of circles being as their circumferences, these will also be in arithmetical progression. But, in such a progression, the sum of the extremes is equal to the sum of each pair of terms equally distant from them; therefore the sum of the circumferences on AC and CB, is equal to the sum of those on AD and DB, and of those on AE and EB, &c, and each sum equal to the semi-circumference of the given circle on the diameter AB. Therefore all the parts have equal perimeters; and each is equal to the whole circumference of the proposed circle. Which satisfies one of the conditions in the problem.

"Again, the same diameters being as the numbers 1, 2, 3, 4, &c, and the areas of circles being as the squares of their diameters, the semicircles will be as the square numbers 1, 4, 9, 16, &c, and consequently the differences between all the adjacent semicircles are as the terms of the arithmetical progression 1, 3, 5, 7, &c; and here again the sums of the extremes, and of every two equidistant means, make up the several equal parts of the circle. Which is the other condition."

3. But this subject admits of a more geometrical form, and is capable of being rendered very general and extensive, and is moreover very fruitful in curious consequences. For first, in whatever ratio the whole diameter is divided, whether into equal or unequal parts, and whatever be the number of parts, the perimeters of the spaces will always be equal. For since the circumferences of circles are in the same ratio as their diameters, and because AB and AD + DB and AC + CB are all equal, therefore the semi-circumferences c and $b + d$ and $a + e$ are all equal, and constant, by the same, whatever be the ratio of the parts AD, DC, CB, of the diameter. We shall presently find too that the spaces TV, RS, and PQ, will be universally as the same parts AD, DC, CB, of the diameter.



4. The semicircles having been described as before mentioned, erect CE perpendicular to AB, and join BE. Then will the circle on the diameter BE, be equal to the space PQ. For, join AE.

Now the space P = semicircle on AB − semicircle on AC;
but the semicir. on AB = semicir. on AE + semicir. on BE,
and the semicir. on AC = semicir. on AE − semicir. on CE,
theref. semic. AB − semic. AC = semic. BE + semicir. CE,
that is, the space P is = semic. BE + semicir. CE;
to each of these add the space Q, or the semicircle on BC,
then P + Q = semic. BE + semic. CE + semic. BC,
that is P + Q = double the semic. BE, or = the whole circle on BE.

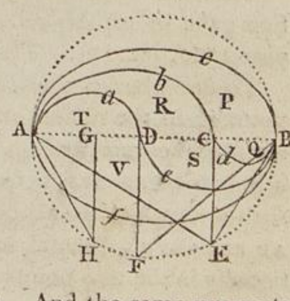
5. In like manner, the two spaces PQ and RS together, or the whole space PQRS, is equal to the circle on the diameter BF. And therefore the space RS alone, is equal to the difference, or the circle on BF *minus* the circle on BE.

6. But, circles being as the squares of their diameters, BE^2, BF^2, and these again being as the parts or lines BC, BD,
therefore the spaces PQ, PQRS, RS, TV,
are respectively as the lines BC, BD, CD, AD.
And if BC be equal to CD, then will PQ be equal to RS, as in the first or simplest case.

7. Hence, to find a circle equal to the space RS, where the points D and C are taken at random: From either end of the diameter, as A, take AG equal to DC, erect GH perpendicular to AB, and join AH; then the circle on AH will be equal to the space RS. For, the space PQ : the space RS :: BC : CD or AG, that is as BE^2 : AH^2 the squares of the diameters, or as the circle on BE to the circle on AH; but the circle on BE is equal to the space PQ, and therefore the circle on AH is equal to the space RS.

8. Hence, to divide a circle in this manner, into any proposed number of parts, that shall be in any ratios to one another: Divide the diameter into as many parts, at the points D, C, &c, and in the same ratios as those proposed; then on the several distances of these points, from the two ends A and B, as diameters, describe the alternate semicircles on the different sides of the whole diameter AB: and they will divide the whole circle in the manner proposed. That is, the spaces TV, RS, PQ, will be as the lines AD, DC, CB.

9. But these properties are not confined to the circle alone. They are found also in the ellipse, as the genus of which the circle is only a species. For if the annexed figure be an ellipse described on the axis AB, the area of which is, in like manner, divided by similar semi-ellipses, described on AD, AC, BC, BD, as axes, all the semi-perimeters *f*, *ae*, *bd*, *c*, will be equal to one another, for the same reason as before in Art. 3, namely, because the peripheries of ellipses are as their diameters. And the same property



would still hold good, if AB were any other diameter of the ellipse, instead of the axis; describing on the parts of it semi-ellipses which shall be similar to those into which the diameter AB divides the given ellipse.

10. And further, if a circle be described about the ellipse, on the diameter AB, and lines be drawn similar to those in the second figure; then, by a process the very same as in Art. 4, *et seq.* substituting only semiellipse for semicircle, it is found that the space

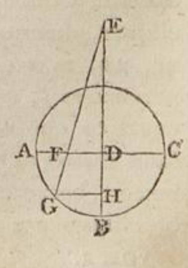
PQ is equal to the similar ellipse on the diameter BE,
PQRS is equal to the similar ellipse on the diameter BF,
RS is equal to the similar ellipse on the diameter AH,
or to the difference of the ellipses on BF and BE;
also the elliptic spaces - - - PQ, PQRS, RS, TV,
are respectively as the lines - BC, BD, DC, AD,

the same ratio as the circular spaces. And hence an ellipse is divided into any number of parts, in any assigned ratios, in the same manner as the circle is divided, namely, dividing the axis, or any diameter in the same manner, and on the parts of it describing similar semiellipses.

TRACT XV.

AN APPROXIMATE GEOMETRICAL DIVISION OF THE CIRCLE.

THE solution, here improved, of the following problem, I first gave in my Miscellanea Mathematica, published in the year 1775, pa. 311. The problem is as follows.



To find whether there is any such fixed point E, in the radius BD produced, bisecting the semicircle ABC, so that any line EFG being drawn from it, this line shall always cut the perpendicular radius AD and the quadrantal arc AB, proportionally in the two points F and G; viz. so that DF shall be to BG in a constant ratio.