TRACT XIII.

ON THE COMMON SECTIONS OF THE SPHERE AND CONE. WITH THE DEMONSTRATION OF SOME OTHER NEW PROPERTIES OF THE SPHERE, WHICH ARE SIMILAR TO CERTAIN KNOWN PROPERTIES OF THE CIRCLE.

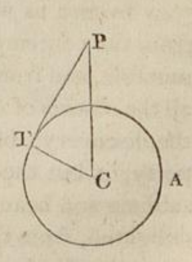
THE study of the mathematical sciences is useful and profitable, not only on account of the benefit derivable from them to the affairs of mankind in general; but are most eminently so, for the pleasure and delight which the human mind feels in the discovery and contemplation of the endless number of truths, that are continually presenting themselves to our view. These meditations are of a sublimity far above all others, whether they be purely intellectual, or whether they respect the nature and properties of material objects; they methodize, strengthen, and extend the reasoning faculties in the most eminent degree, and so fit the mind the better for understanding and improving every other science; but, above all, they furnish us with the purest and most permanent delight, from the contemplation of truths peculiarly certain and immutable, and from the beautiful analogy which reigns through all the objects of similar inquiry. In the mathematical sciences, the discovery, often accidental, of a plain and simple property, is but the harbinger of a thousand others of the most sublime and beautiful nature, to which we are gradually led, delighted, from the more simple, to the more compound and general, till the mind becomes quite enraptured at the full blaze of light bursting upon it from all directions.

Of these very pleasing subjects, the striking analogy that prevails among the properties of geometrical figures, or figured extension, is not one of the least. Here we often find that a plain and obvious property of one of the simplest figures, leads us to, and forms only a particular case of, a property in some other figure, less simple; afterwards this again turns out to be no more than a particular case of another still more general; and so on, till at last we often trace the tendency to end in a general property of all figures whatever.

The few properties which make a part of this paper, constitute a small specimen of the analogy, and even identity, of some of the more remarkable properties of the circle, with those of the sphere. To which are added some properties of the lines of section, and of contact, between the sphere and cone. Both which may be further extended as occasions may offer: like as all of these properties have occurred from the circumstance, mentioned near the end of the paper, of considering the inner surface of a hollow spherical vessel, as viewed by an eye, or as illuminated by rays, from a given point.

PROPOSITION 1.

All the tangents are equal, which are drawn, from a given point without a sphere, to the surface of the sphere, quite around.



Demons.—For, let PT be any tangent from the given point P; and draw PC to the centre C, and join TC. Also let CTA be a great circle of the sphere in the plane of the triangle TPC. Then, CP and CT, as well as the angle T, which is right (Eucl. iii. 18), being constant, in every position of the tangent, or of the point of contact T; the square of PT will be every where equal to the difference of the squares of the constant lines CP, CT, and therefore constant; and consequently the line or tangent PT itself of a constant length, in every position, quite round the surface of the sphere.

PROP. 2.

If a tangent be drawn to a sphere, and a radius be drawn from the centre to the point of contact, it will be perpendicular to the tangent; and a perpendicular to the tangent will pass through the centre.

Demons.—For, let PT be the tangent, TC the radius, and CTA a great circle of the sphere, in the plane of the triangle TPC, as in the foregoing proposition. Then, PT touching the circle in the point T, the radius TC is perpendicular to the tangent PT, by Eucl. iii. 18, 19.

PROP. 3.

If any line or chord be drawn in a sphere, its extremes terminating in the circumference; then a perpendicular drawn to it, from the centre, will bisect it: and if the line drawn from the centre, bisect it, it is perpendicular to it.

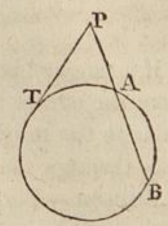
Demons.—For, a plane may pass through the given line and the centre of the sphere; and the section of that plane with the sphere, will be a great circle (Theodos. i. 1), of which the given line will be a chord. Therefore (Eucl. iii. 3) the perpendicular bisects the chord, and the bisecting line is perpendicular.

Corol.—A line drawn from the centre of the sphere, to the centre of any lesser circle, or circular section, is perpendicular to the plane of that circle. For, by the proposition, it is perpendicular to all the diameters of that circle.

PROP. 4.

If from a given point, a right line be drawn in any position through a sphere, cutting its surface always in two points; the rectangle contained under the whole line and the external part, that is the rectangle contained by the two distances between the given point, and the two points where the line meets the surface of the sphere, will always be of the same constant magnitude, namely, equal to the square of the tangent drawn from the same given point.

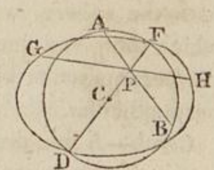
Demons.—Let P be the given point, and AB the two points in which the line PAB meets the surface of the sphere; through PAB and the centre let a plane cut the sphere in the great circle TAB, to which draw the tangent PT. Then the rectangle PA . PB is equal to the square of PT (Eucl. iii. 36); but PT, and consequently its square, is constant by Prop. 1; therefore the rectangle PA . PB, which is always equal to this square, is every where of the same constant magnitude.



PROP. 5.

If any two lines intersect each other within a sphere, and be terminated at the surface on both sides; the rectangle of the parts of the one line, will be equal to the rectangle of the parts of the other. And, universally, the rectangles of the two parts of all lines passing through the point of intersection, are all of the same magnitude.

Demons.—Through any one of the lines, as AB, conceive a plane to be drawn through the centre C of the sphere, cutting the sphere in the great circle ABD; and draw its diameter DCPF through the points of intersection P of all the lines. Then the rectangle AP . PB is equal to the rectangle DP . PL (Eucl. iii. 35).



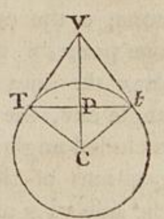
Again, through any other of the intersecting lines GH, and the centre, conceive another plane to pass, cutting the sphere in another great circle DGFH. Then, because the points C and P are in this latter plane, the line CP, and consequently the whole diameter DCPF, is in the same plane; and therefore it is a diameter of the circle DGFH, of which GPH is a chord. Therefore, again, the rectangle GP . PH is equal to the rectangle DP . PF (Eucl. iii. 35).

Consequently all the rectangles AP . PB, GP . PH, &c, are equal, being each equal to the constant rectangle DP . PF.

Corol.—The great circles passing through all the lines or chords which intersect in the point P, will all intersect in the common diameter DPF.

PROP. 6.

If a sphere be placed within a cone, so as to touch it in two points; then shall the outside of the sphere, and the inside of the cone, mutually touch quite around, and the line of contact will be a circle.



Demons.—Let V be the vertex of the cone, C the centre of the sphere, T one of the two points of contact, and TV a side of the cone. Draw CT, CV. Then TVC is a triangle right-angled at T (Prop. 2). In like manner, *t* being another point of contact, and C*t* being drawn, the triangle *t*VC will be right-angled at *t*. These two triangles then, TVC, *t*VC, having the two sides CT, TV, equal to the two C*t*, *t*V (Prop. 1), and the included angle T equal to the included angle *t*, will be equal in all respects (Eucl. i. 4), and consequently have the angle TVC equal to the angle *t*VC.

Again, let fall the perpendiculars TP, *t*P. Then the two triangles TVP, *t*VP, having the two angles TVP and TPV equal to the two *t*VP and *t*PV, and the side TV equal to the side *t*V (Prop. 1), will be equal in all respects (Eucl. i. 26); consequently TP is equal to *t*P, and VP equal to VP. Hence PT, P*t* are radii of a little circle of the sphere, whose plane is perpendicular to the line CV, and its circumference every where equidistant from the point C or V. This circle is therefore a circular section both of the sphere and of the cone, and is therefore the line of their mutual contact. Also CV is the axis of the cone.

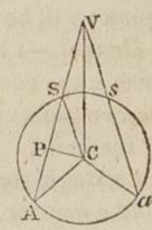
Corol. 1.—The axis of a cone, when produced, passes through the centre of the inscribed sphere.

Corol. 2.—Hence also, every cone circumscribing a sphere, so that their surfaces touch quite around, is a right cone;

nor can any scalene or oblique cone touch a sphere in that manner.

PROP. 7.

The two common sections of the surfaces of a sphere and a right cone, are the circumferences of circles, if the axis of the cone pass through the centre of the sphere.



Demons.—Let V be the vertex of the cone, C the centre of the sphere, and S one point of the less or nearer section; draw the lines CS, CV. Then, in the triangle CSV, the two sides CS, CV, and the included angles CV, are constant, for all positions of the side VS; and therefore the side VS is of a constant length for all positions, and is consequently the side of a right cone having a circular base; therefore the locus of all the points S, is the circumference of a circle perpendicular to the axis CV, that is, the common section of the surfaces of the sphere and cone, is that circumference.

In the same manner it is proved that, if A be any point in the farther or greater section, and CA be drawn; then VA is constant for all positions, and therefore, as before, is the side of a cone cut off by a circular section whose plane is perpendicular to the axis.

And these circles, being both perpendicular to the axis, are parallel to each other. Or, they are parallel because they are both circular sections of the cone.

Corol. 1.—Hence SA = *sa*, because VA = V*a*, and VS = V*s*.

Corol. 2.—All the intercepted equal parts SA, *sa*, &c, are equally distant from the centre. For, all the sides of the triangle SCA are constant, and therefore the perpendicular CP is constant also. And thus all the equal right lines or chords in a sphere, are equally distant from the centre.

Corol. 3.—The sections are not circles, and therefore not in planes, if the axis pass not through the centre. For then some of the points of section are farther from the vertex than others.

PROP. 8.

Of the two common sections of a sphere and an oblique cone, if the one be a circle, the other will be a circle also.

Demons.—Let SA*as* and ASV*a* be sections of the sphere and cone, made by a common plane passing through the axes of the cone and the sphere; also S*s*, A*a* the diameters of the two sections. Now, by the supposition, one of these, as A*a*, is the diameter of a circle. But the angle VS*s* = the angle V*a*A (Eucl. i. 13, and iii. 22), therefore S*s* cuts the cone in sub-contrary position to A*a*; and consequently, if a plane pass through S*s*, and perpendicular to the plane AV*a*, its section with the oblique cone will be a circle, whose diameter is the line S*s* (Apol. i. 5). But the section of the same plane and the sphere, is also a circle whose diameter is the same line S*s* (Theod. i. 1). Consequently the circumference of the same circle, whose diameter is S*s*, is in the surface both of the cone and sphere; and therefore that circle is the common section of the cone and sphere.

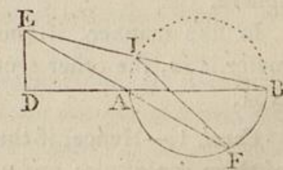
In like manner, if the one section be a circle whose diameter is S*a*, the other section will be a circle whose diameter is *s*A.

Corol. 1.—Hence, if the one section be not a circle, neither of them is a circle; and consequently they are not in planes; for the section of a sphere by a plane, is a circle.

Corol. 2.—When the sections of a sphere and oblique cone are circles, the axis of the cone does not pass through the centre of the sphere, (except when one of the sections is a great circle, or passes through the centre). For, the axis passes through the centre of the base, but not perpendicularly; whereas a line drawn from the centre of the sphere to the centre of the base, is perpendicular to the base, by cor. to prop. 3.

Corol. 3.—Hence, if the inside of a bowl, which is a hemisphere, or any segment of the sphere, be viewed by an eye not situated in the axis produced, which is perpendicular to the section or brim; the lower, or extreme part of the internal surface which is visible, will be bounded by a circle of the sphere; and the part of the surface seen by the eye, will be included between the said circle, and border or brim, which it intersects in two points. For the eye is in the place of the vertex of the cone; and the rays from the eye to the brim of the bowl, and thence continued from the nearer part of the brim, to the opposite internal surface, form the sides of the cone; which, by the proposition, will form a circular arc on the said internal surface; because the brim, which is the one section, is a circle.

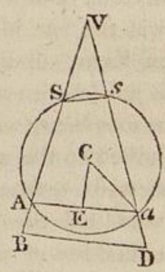
And hence, the place of the eye being given, the quantity of internal surface that can be seen, may be easily determined. For the distance and height of the eye, with respect to the brim, will give the greatest distance of the section below the brim, together with its magnitude and inclination to the plane of the brim; which being known, common mensuration furnishes us with the measure of the surface included between them. Thus, if AB be the diameter in the vertical plane passing through the eye at E, also AFB the section of the bowl by the same plane, and AIB the supplement of that arc. Draw EAF, EIB, cutting this vertical circle in F and I; and join IF. Then shall IF be the diameter of the section or extremity of the visible surface, and BF its greatest distance below the brim, an arc which measures an angle double the angle at A.



Corol. 4.—Hence also, and from Proposition 4, it follows, that if through every point in the circumference of a circle, lines be drawn to a given point E out of the plane of the circle, so that the rectangle contained under the parts between the point E and the circle, and between the same point E and some other point F, may always be of a certain given magnitude; then the locus of all the points F will also be a circle, cutting the former circle in the two points where the lines drawn from the given point E, to the several points in the circumference of the first circle, change from the convex to the concave side of the circumference. And the constant quantity, to which the rectangle of the parts is always equal, is equal to the square of the line drawn from the given point E to either of the said two points of intersection.—And thus the loci of the extremes of all such lines, are circles.

PROP. 9.

Prob.—To place a given sphere, and a given oblique cone, in such positions, that their mutual sections shall be circles.



Let V be the vertex, VB the least side, and VD the greatest side of the cone. In the plane of the triangle VBD it is evident will be found the centre of the sphere. Parallel to BD draw A*a* the diameter of a circular section of the cone, so that it be not greater than the diameter of the sphere. Bisect A*a* with the perpendicular EC; with the centre A and radius of the sphere, cut EC in C, which will be the centre of the sphere; from which therefore describe a great circle of it, cutting the sides of the cone in the points S, *s*, A, *a*: so shall S*s* and A*a* be the diameters of circular sections which are common to both the sphere and cone.

July 29, 1785.