26. For equations of higher dimensions, as the 5th, the 6th, the 7th, &c, we might, in imitation of this last method, combine other forms of quantities together. Thus, for the 5th power, we might compare it either with $(x-a)^4 \times (x-b)$, or with $(x-a)^3 \times (x-b)^2$, or with $(x-a)^3 \times (x-b) \times (x-c)$, or with $(x-a)^2 \times (x-b)^2 \times (x-c)$. And so for the other powers.

TRACT XII.

OF THE BINOMIAL THEOREM. WITH A DEMONSTRATION OF THE TRUTH OF IT IN THE GENERAL CASE OF FRACTIONAL EXPONENTS.

1. It is well known that this celebrated theorem is called *binomial*, because it contains a proposition of a quantity consisting of *two* terms, as a radix, to be expanded in a series of equal value. It is also called emphatically the Newtonian theorem, or Newton's binomial theorem, because he has commonly been reputed the author of it, as he was indeed for the case of fractional exponents, which is the most general of all, and includes all the other particular cases, of powers, or divisions, &c.

2. The binomial, as proposed in its general form, was, by Newton, thus expressed $\overline{P + PQ}^{\frac{m}{n}}$; where P is the first term of the binomial, Q the quotient of the second term divided by the first, and consequently PQ is the second term itself; or PQ may represent all the terms of a multinomial, after the first term, and consequently Q the quotient of all those terms, except the first term, divided by that first term, and may be either positive or negative; also $\frac{m}{n}$ represents the exponent of the binomial, and may denote any quantity, integral or

fractional, positive or negative, rational or surd. When the exponent is integral, the denominator n is equal to 1, and the quantity then in this form $(P + PQ)^m$, denotes a binomial to be raised to some power; the series for which was fully determined before Newton's time, as will be shown in the course of the 19th Tract of this volume. When the exponent is fractional, m and n may be any quantities whatever, m denoting the index of some power to which the binomial is to be raised, and n the index of the root to be extracted of that power: and to this case it was first extended and applied by Newton. When the exponent is negative, the reciprocal of the same quantity is meant; as

$(P+PQ)^{-\frac{m}{n}}$ is equal to $\dfrac{1}{(P + PQ)^{\frac{m}{n}}}$.

3. Now when the radical binomial is expanded in an equivalent series, it is asserted that it will be in this general form, namely $(P + PQ)^{\frac{m}{n}}$ or $P^{\frac{m}{n}} \times (1 + Q)^{\frac{m}{n}} =$

$$P^{\frac{m}{n}} \times 1 + \frac{m}{n}Q + \frac{m}{n} \cdot \frac{m-n}{2n}Q^2 + \frac{m}{n} \cdot \frac{m-n}{2n} \cdot \frac{m-2n}{3n}Q^3 + \&c),$$

$$\text{or } P^{\frac{m}{n}} \times 1 + \frac{m}{n}AG + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + \&c.$$

where the law of the progression is visible, and the quantities P, m, n, Q, include their signs + or −, the terms of the series being all positive when Q is positive, and alternately positive and negative when Q is negative, independent however of the effect of the coefficients made up of m and n: also A, B, C, D, &c, in the latter form, denote each preceding term. This latter form is the easier in practice, when we want to collect the sum of the terms of a series; but the former is the fitter for showing the law of the progression of the terms.

4. The truth of this series was not demonstrated by Newton, but only inferred by way of induction. Since his time however, several attempts have been made to demonstrate it, with various success, and in various ways; of which however

those are justly preferred, which proceed by pure algebra, and without the help of fluxions. And such has been esteemed the difficulty of proving the general case, independent of the doctrine of fluxions, that many eminent mathematicians to this day account the demonstration not fully accomplished, and still a thing greatly to be desired. Such a demonstration I think is here effected. But before delivering it, it may not be improper to premise somewhat of the history of this theorem, its rise, progress, extension, and demonstrations.

5. Till very lately the prevailing opinion has been, that the theorem was not only invented by Newton, but first of all by him; that is, in that state of perfection in which the terms of the series, for any assigned power whatever, can be found independently of the terms of the preceding powers; namely, the second term from the first, the third term from the second, the fourth term from the third, and so on, by a general rule. Upon this point I have already given an opinion in the history to my logarithms, above cited, and I shall here enlarge somewhat further on the same head.

That Newton invented it himself, I make no doubt. But that he was not the first inventor, is at least as certain. It was described by Briggs, in his Trigonometria Britannica, long before Newton was born; not indeed for fractional exponents, for that was the application of Newton, but for any integral power whatever, and that by the general law of the terms as laid down by Newton, independent of the terms of the powers preceding that which is required. For as to the generation of the coefficients of the terms of one power from those of the preceding powers, successively one after another, it was remarked by Vieta, Oughtred, and many others, and was not unknown to much more early writers on arithmetic and algebra, as will be manifest by a slight inspection of their works, as well as the gradual advance the property made, both in extent and perspicuity, under the hands of the successive masters in arithmetic, every one adding somewhat more towards the perfection of it.

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6. Now the knowledge of this property of the coefficients of the terms in the powers of a binomial, is at least as old as the practice of the extraction of roots; for this property was both the foundation and the principle, as well as the means of those extractions. And as the writers on arithmetic became acquainted with the nature of the coefficients in powers still higher, just so much higher did they extend the extraction of roots, still making use of this property. At first it seems they were only acquainted with the nature of the square, which consists of these three terms, 1, 2, 1; and accordingly they extracted the square roots of numbers by means of them; but went no further. The nature of the cube next presented itself, which consists of these four terms, 1, 3, 3, 1; and by means of these they extracted the cubic roots of numbers, in the same manner as we do at present. And this was the extent of their extractions in the time of Lucas de Burgo, an Italian, who, from 1470 to 1500, wrote several tracts on arithmetic, containing the sum of what was then known of this science, which chiefly consisted in the doctrine of the proportions of numbers, the nature of figurate numbers, and the extraction of roots, as far as the cubic root inclusively.

7. It was not long however before the nature of the coefficients of all the higher powers became known, and tables formed for constructing them indefinitely. For in the year 1544 came out, at Norimberg, an excellent treatise of arithmetic and algebra, by Michael Stifelius, a German divine, and an honest, but a weak, disciple of Luther. In this work, *Arithmetica Integra*, of Stifelius, are contained several curious things, some of which have been ascribed to a much later date. He here treats, pretty fully and ably, of progressional and figurate numbers, and in particular of the following table for constructing both them and the coefficients of the terms of all powers of a binomial, which has been so often used since his time for these and other purposes, and which more than a century after was, by Pascal, otherwise called the

arithmetical triangle, but who only mentioned some additional properties of the table.

1							
2							
3	3						
4	6						
5	10	10					
6	15	20					
7	21	35	35				
8	28	56	70				
9	36	84	126	126			
10	45	120	210	252			
11	55	165	330	462	462		
12	66	220	495	792	924		
13	78	286	715	1287	1716	1716	
14	91	364	1001	2002	3003	3432	
15	105	455	1365	3003	5005	6435	6435
16	120	560	1820	4368	8008	11440	12870
17	136	680	2380	6188	12376	19448	24310

Stifelius here observes that the horizontal lines of this table furnish the coefficients of the terms of the correspondent powers of a binomial; and teaches how to use them in extracting the roots of all powers whatever. And after him the same table was used for the same purpose, by Cardan, and Stevin, and the other writers on arithmetic. I suspect however, that the nature of this table was known much earlier than the time of Stifelius, at least so far as regards the progressions of figurate numbers, a doctrine amply treated of by Nichomachus, who lived, according to some, before Euclid, but not till long after him according to others. His work on arithmetic was published at Paris in 1538; and it is supposed was chiefly copied into the treatise on the same subject by Boethius: but I have never seen either of these two works. Though indeed Cardan seems to ascribe the invention of the table to Stifelius; but I suppose that is only to be understood of its application to the extraction of roots. See Cardan's *Opus Novum de Proportionibus*, where he quotes it, and extracts the table and its use from Stifelius's book. Cardan also, at p. 185, *et seq.* of the same work, makes use of a like table to find the number of variations of things, or conjugations as he calls them.

8. The contemplation of this table has probably been attended with the invention and extension of some of our most curious discoveries in mathematics, both in regard to the powers of a binomial, with the consequent extraction of roots, the doctrine of angular sections by Vieta, and the differential method by Briggs and others. For, one or two of the powers or sections being once known, the table would be of excellent use in discovering and constructing the rest. And accordingly we find this table used on many occasions by Stifelius, Cardan, Stevin, Vieta, Briggs, Oughtred, Mercator, Pascal, &c, &c.

9. On this occasion I cannot help mentioning the ample manner in which I see Stifelius, at fol. 35, *et seq.* of the same book, treats of the nature and use of logarithms, though not under the same name, but under the idea of a series of arithmeticals, adapted to a series of geometricals. He there explains all their uses; such as, that the addition of them, answers to the multiplication of their geometricals; subtraction to division; multiplication of exponents, to involution; and dividing of exponents, to evolution. And he exemplifies the use of them in cases of the Rule-of-Three, and in finding mean proportionals between given terms, and such like, exactly as is done in logarithms. So that he seems to have been in the full possession of the idea of logarithms, and wanted only the necessity of troublesome calculations to induce him to make a table of such numbers.

10. But though the nature and construction of this table, namely of figurate numbers, was thus early known, and employed in the raising of powers, and extracting of roots; yet it was only by raising the numbers one from another by continual additions, and then taking them from the table for use when wanted; till Briggs first pointed out the way of raising any horizontal line in the foregoing table by itself, without any of the preceding lines; and thus teaching to raise the terms of any power of a binomial, independent of any other

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powers; and so gave the substance of the binomial series in words, wanting only the notation in symbols; as it is shown at large in the 19th Tract, in this volume.

11. Whatever was known however of this matter, related only to pure or integral powers, no one before Newton having thought of extracting roots by infinite series. He happily discovered, that, by considering powers and roots in a continued series, roots being as powers having fractional exponents, the same binomial series would equally serve for them all, whether the index should be fractional or integral, or the series be finite or infinite.

12. The truth of this method however was long known only by trial in particular cases, and by induction from analogy. Nor does it appear that even Newton himself ever attempted any direct proof of it. But various demonstrations of this theorem have been since given by the more modern mathematicians, of which some are by means of the doctrine of fluxions, and others, more legally, from the pure principles of algebra only. Some of which I shall here give a short account of.

13. One of the first demonstraters of this theorem, was Mr. James Bernoulli. His demonstration is, among several other curious things, contained in this little work called *Ars Conjectandi*, which has been improperly omitted in the collection of his works published by his nephew Nicholas Bernoulli. This is a strict demonstration of the binomial theorem in the case of integral and affirmative powers, and is to this effect. Supposing the theorem to be true in any one power, as for instance, in the cube, it must be true in the next higher power; which he demonstrates. But it is true in the cube, in the fourth, fifth, sixth, and seventh powers, as will easily appear by trial, that is by actually raising those powers by continual multiplications. Therefore it is true in all higher powers. All this he shows in a regular and legitimate manner, from the principles of multiplication, and without the

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help of fluxions. But he could not extend his proof to the other cases of the binomial theorem, in which the powers are fractional. And this demonstration has been copied by Mr. John Stewart, in his commentary on Newton's quadrature of curves. To which he has added, from the principles of fluxions, a demonstration of the other case, for roots or fractional exponents.

14. In No. 230 of the Philosophical Transactions for the year 1697, is given a theorem, by Mr. De Moivre, in imitation of the binomial theorem, which is extended to any number of terms, and thence called the multinomial theorem; which is a general expression in a series, for raising any multinomial quantity to any power. His demonstration of the truth of this theorem, is independent of the truth of the binomial theorem, and contains in it a demonstration of the binomial theorem as a subordinate proposition, or particular case of the other more general theorem. And this demonstration may be considered as a legitimate one, for pure powers, founded on the principles of multiplication, that is, on the doctrine of combinations and permutations. And it proves that the law of the continuation of the terms, must be the same in the terms not computed, or not set down, as in those that are written down.

15. The ingenious Mr. Landen has given an investigation of the binomial theorem, in his *Discourse concerning the Residual Analysis*, printed in 1758, and in the *Residual Analysis* itself, printed in 1764. The investigation is deduced from this lemma, namely, if m and n be any integers, and $q = \frac{v}{x}$, then is

$$\frac{x^{\frac{m}{n}} - v^{\frac{m}{n}}}{x - v} = x^{\frac{m}{n}-1} \times \frac{1 + q + q^2 + q^3 \; - \; - \; (m)}{1 + q^{\frac{m}{n}} + q^{\frac{2m}{n}} + q^{\frac{3m}{n}} \; - \; - \; (n)}$$

which theorem is made the principal basis of his Residual Analysis.

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The investigation is thus: the binomial proposed being $(1+x)^{\frac{m}{n}}$, assume it equal to the following series $1+ax+bx^2+cx^3$ &c, with indeterminate coefficients. Then for the same reason

$$\text{as } (1+x)^{\frac{m}{n}} \text{ is } = 1+ax+bx^2+cx^3 \text{ \&c,}$$

$$\text{will } (1+y)^{\frac{m}{n}} \text{ be } = 1+ay+by^2+cy^3 \text{ \&c.}$$

Then, by subtraction,

$$(1+x)^{\frac{m}{n}}-(1+y)^{\frac{m}{n}} = a(x-y)+b(x^2-y^2)+c(x^3-y^3) \text{ \&c.}$$

And, dividing both sides by $x-y$, and by the lemma, we have

$$\frac{(1+x)^{\frac{m}{n}}-(1+y)^{\frac{m}{n}}}{x-y} = (1+x)^{\frac{m}{n}-1} \times \frac{1+\frac{1+y}{1+x}+(\frac{1+y}{1+x})^2+(\frac{1+y}{1+x})^3 \text{ - - - } (m)}{1+(\frac{1+y}{1+x})^{\frac{m}{n}}+(\frac{1+y}{1+x})^{\frac{2m}{n}}+(\frac{1+y}{1+x})^{\frac{3m}{n}} \text{ - - - } (n)}$$

$$= a+b(x+y)+c(x^2+xy+y^2)+d(x^3+x^2y+xy^2+y^3) \text{ \&c.}$$

Then, as this equation must hold true whatever be the value of y, take $y=x$, and it will become

$$\frac{m}{n} \times (1+x)^{\frac{m}{n}-1} = a+2bx+3cx^2+4cx^3 \text{ \&c.}$$

Consequently, multiplying by $1+x$, we have

$\frac{m}{n} \times (1+x)^{\frac{m}{n}}$, or its equal by the assumption,

$$\text{viz. } \frac{m}{n}+\frac{m}{n}ax+\frac{m}{n}bx^2+\frac{m}{n}cx^3 \text{ \&c.}$$

$$= a + \left.\begin{matrix}2b\\a\end{matrix}\right\} x + \left.\begin{matrix}3c\\2b\end{matrix}\right\} x^2 + \left.\begin{matrix}4d\\3c\end{matrix}\right\} x^3 \text{ \&c.}$$

Then, by comparing the homologous terms, the value of the coefficients a, b, c, &c, are deduced for as many terms as are compared.

A large account is also given of this investigation by the learned Dr. Hales, in his *Analysis Equationum*, lately published at Dublin.

Mr. Landen then contrasts this investigation with that by the method of fluxions, which is as follows. Assume as before;

$$(1+x)^{\frac{m}{n}} = 1 + ax + bx^2 + cx^3 + dx^4 \text{ \&c.}$$

Take the fluxion of each side, and we have

$$\frac{m}{n} \times (1+x)^{\frac{m}{n}-1} \times \dot{x} = a\dot{x} + 2bx\dot{x} + 3cx^2\dot{x} \text{ \&c.}$$

Divide by $\dot{x}$, or take it $= 1$, so shall

$$\frac{m}{n} \times (1+x)^{\frac{m}{n}-1} = a + 2bx + 3cx^2 + 4dx^3 \text{ \&c.}$$

Then multiply by $1 + x$, and so on as above in the other way.

16. Besides the above, and an investigation by the celebrated M. Euler, which are the principal demonstrations and investigations that have been given of this important theorem, I have been shown an ingenious attempt of Mr. Baron Maseres, to demonstrate this theorem in the case of roots or fractional exponents, by the help of De Moivre's multinomial theorem. But, not being quite satisfied with his own demonstration, as not expressing the law of continuation of the terms which are not actually set down, he was pleased to urge me to attempt a more complete and satisfactory demonstration of the general case of roots, or fractional exponents. And he further proposed it in this form, namely, that if Q be the coefficient of one of the terms of the series which is equal to $(1+x)^{\frac{1}{n}}$, and P the coefficient of the next preceding term, and R the coefficient of the next following term; then, if Q be $= \frac{a}{b} \times$ P, it is required to prove that R will be $= \frac{a-n}{b+n} \times$ Q. This he observed would be quite perfect and satisfactory, as it would include all the terms of the series, as well those that are omitted, as those that are actually set down. And I was, in my demonstration, to suppose, if I pleased, the truth of the binomial and multinomial theorems for integral powers, as truths that had been previously and perfectly proved.

In consequence I sent him soon after the substance of the following demonstration; with which he was quite satisfied, and which I now proceed to explain at large.

17. Now the binomial integral is $(1+x)^n =$

$$1+\overbrace{\frac{n}{1}}^{a}x+\overbrace{\frac{n}{1}\cdot\frac{n-1}{2}}^{b}x^2+\overbrace{\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}}^{c}x^3+\overbrace{\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}}^{d}x^4\ \&c,$$

$$\text{or } 1+\frac{n}{1}x+\frac{n-1}{2}ax^2+\frac{n-2}{3}bx^3+\frac{n-3}{4}cx^4+\&c,$$

where a, b, c, &c, denote the whole coefficients of the 2d, 3d, 4th, &c, terms, over which they are placed; and in which the law is this, namely, if P, Q, R, be the coefficients of any three terms in succession, and if $\frac{g}{h}$P be = Q, then is $\frac{g-1}{h+1}$Q = R; as is evident; and which, it is granted, has been proved.

18. And the binomial fractional is $(1+x)^{\frac{1}{n}} =$

$$1+\overbrace{\frac{1}{n}}^{a}x+\overbrace{\frac{1}{n}\cdot\frac{1-n}{2n}}^{b}x^2+\overbrace{\frac{1}{n}\cdot\frac{1-n}{2n}\cdot\frac{1-2n}{3n}}^{c}x^3+\overbrace{\frac{1}{n}\cdot\frac{1-n}{2n}\cdot\frac{1-2n}{3n}\cdot\frac{1-3n}{4n}}^{d}x^4$$

$$\&c,\ \text{ or } 1+\frac{1}{n}x+\frac{1-n}{2n}ax^2+\frac{1-2n}{3n}bx^3+\frac{1-3n}{4n}cx^4+\&c;$$

in which the law is this, namely, if P, Q, R be the coefficients of three terms in succession; and if $\frac{g}{h}$P be = Q, then is $\frac{g-n}{h+n}$Q = R. Which is the property to be proved.

19. Again, the multinomial integral $(1 + Ax + Bx^2 + Cx^3 \&c)^n$, is = . . . 1

$$(a) \quad + \frac{n}{1}Ax \qquad + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}A^4x^4$$

$$+ \frac{n}{1}\cdot\frac{n-1}{2}A^2x^2 \qquad + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{1}A^2B$$

$$(b) \quad + \frac{n}{1}B \qquad (d) \quad + \frac{n}{1}\cdot\frac{n-1}{1}AC$$

$$+ \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}A^3x^3 \qquad + \frac{n}{1}\cdot\frac{n-1}{2}B^2$$

$$(c) \quad + \frac{n}{1}\cdot\frac{n-1}{1}AB \qquad + \frac{n}{1}D$$

$$+ \frac{n}{1}C$$

$$+ \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}\cdot\frac{n-4}{5}A^5x^5$$

$$+ \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{1}A^3B$$

$$+ \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{1}A^2C$$

$$(e) \quad + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{2}AB^2$$

$$+ \frac{n}{1}\cdot\frac{n-1}{2}AD$$

$$+ \frac{n}{1}\cdot\frac{n-1}{1}BC$$

$$+ \frac{n}{1}E$$

&c.

Or, if we put a, b, c, d, &c, for the coefficients of the 2d, 3d, 4th, 5th, &c, terms, or powers of x, the last series, by substitution, will be changed into this form,

$$(1 + \text{A}x + \text{B}x^2 + \text{C}x^3 + \&\text{c})^n = \text{ - - - - - - - } 1$$

$$(a) \qquad + \frac{n\text{A}}{1}x$$

$$(b) \qquad + \frac{2n\text{B} + (n-1)\text{A}a}{2}x^2$$

$$(c) \qquad + \frac{3n\text{C} + (2n-1)\text{B}a + (n-2)\text{A}b}{3}x^3$$

$$(d) \qquad + \frac{4n\text{D} + (3n-1)\text{C}a + (2n-2)\text{B}b + (n-3)\text{A}c}{4}x^4$$

$$(e) \qquad + \frac{5n\text{E} + (4n-1)\text{D}a + (3n-2)\text{C}b + (2n-3)\text{B}c + (n-4)\text{A}d}{5}x^5$$

&c.

20. Now, to find the series in Art. 18, assume the proposed binomial equal to a series with indeterminate coefficients, as

$$(1 + x)^{\frac{1}{n}} = 1 + \text{A}x + \text{B}x^2 + \text{C}x^3 + \text{D}x^4 + \&\text{c}.$$

Then raise each side to the n power, so shall

$$1 + x = (1 + \text{A}x + \text{B}x^2 + \text{C}x^3 + \&\text{c})^n.$$

But it is granted that the multinomial raised to any integral power is proved, and known to be, as in the last Art. viz,

$$1 + x = (1 + \text{A}x + \text{B}x^2 + \text{C}x^3 + \&\text{c})^n =$$

$$1 + \overbrace{\frac{n\text{A}}{1}}^{a}x + \overbrace{\frac{2n\text{B} + (n-1)\text{A}a}{2}}^{b}x^2 + \overbrace{\frac{3n\text{C} + (2n-1)\text{B}a + (n-2)\text{A}b}{3}}^{c}x^3$$

&c.

It follows then, that if this last series be equal to $1 + x$, by equating the homologous coefficients, all the terms after the second must vanish, or all the coefficients b, c, d, &c, after the second term, must be each $= 0$. Writing therefore, in this series, 0 for each of the letters b, c, d, &c, it will become of this more simple form, viz, $1 + x =$

$$1 + \overbrace{\frac{n\text{A}}{1}}^{a}x + \overbrace{\frac{2n\text{B} + (n-1)\text{A}a}{2}}^{b = 0}x^2 + \overbrace{\frac{3n\text{C} + (2n-1)\text{B}a}{3}}^{c = 0}x^3 + \&\text{c}.$$

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Put now each of the coefficients, after the second term, $= 0$, and we shall have these equations

$$2n\text{B} + (1n - 1)\,\text{A}a = 0$$
$$3n\text{C} + (2n - 1)\,\text{B}a = 0$$
$$4n\text{D} + (3n - 1)\,\text{C}a = 0$$
$$5n\text{E} + (4n - 1)\,\text{D}a = 0$$
&c.

The resolution of which equations gives the following values of the assumed indeterminate coefficients, namely,

$$\text{B} = \frac{1-n}{2n}\text{A}a,\ \text{C} = \frac{1-2n}{3n}\text{B}a,\ \text{D} = \frac{1-3n}{4n}\text{C}a,\ \text{E} = \frac{1-4n}{5n}\text{D}a,\ \&\text{c};$$

which coefficients are according to the law proposed, namely, when $\frac{g}{h}\text{P}$ is $= \text{Q}$, then is $\frac{g-n}{h+n}\text{Q} = \text{R}$. Q. E. D.

21. Also, by equating the second coefficients, namely, $1 = a = n\text{A}$, we find $\text{A} = \frac{1}{n}$. This being written for A in the above values of B, C, D, &c, will give the proper series for the binomial in question, namely, $(1 + x)^{\frac{1}{n}}$

$$= 1 + \text{A}x + \text{B}x^2 + \text{C}x^3 + \&\text{c},$$
$$= 1 + \frac{1}{n}x + \frac{1-n}{2n}ax^2 + \frac{1-2n}{3n}bx^3 + \&\text{c},$$
$$= 1 + \frac{1}{n}x + \frac{1}{n}\cdot\frac{1-n}{2n}x^2 + \frac{1}{n}\cdot\frac{1-n}{2n}\cdot\frac{1-2n}{3n}x^3 + \&\text{c}.$$

Of the Form of the Assumed Series.

22. In the demonstrations or investigations of the truth of the binomial theorem, the butt or object has always been the law of the coefficients of the terms: the form of the series, as to the powers of x, having never been disputed, but taken for granted, either as incapable of receiving a demonstration, or as too evident to need one. But since the demonstration of the law of the coefficients has been accomplished, in which the main, if not the only, difficulty was supposed to consist, we have extended our researches still further, and have even doubted or queried the very *form* of the terms themselves,

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namely, $1 + Ax + Bx^2 + Cx^3 + Dx^4 +$ &c, increasing by the regular integral series of the powers of x, as assumed to denote the quantity $(1 + x)^{\frac{1}{n}}$, or the n root of $1 + x$. And in consequence of these scruples, I have been required, by a learned friend, to vindicate the propriety of that assumption. Which I think is effectually done as follows.

23. To prove then, that any root of the binomial $1 + x$ *can* be represented by a series of this form $1 + x + x^2 + x^3 + x^4$ &c, where the coefficients are omitted, our attention being now employed only on the powers of x; let the series representing the value of $(1 + x)^{\frac{1}{n}}$ be $1 + A + B + C + D +$ &c; where A, B, C, &c, now represent the whole of the 2d, 3d, 4th, &c, terms, both their coefficients and the powers of x, whatever they may be, only increasing from the less to the greater, because they increase in the terms $1 + x$ of the given binomial itself; and in which the first term must evidently be 1, the same as in the given binomial.

Raise now $(1 + x)^{\frac{1}{n}}$, and its equivalent series $1 + A + B + C +$ &c, both to the n power, by the multinomial theorem, and we shall have, as before,

$$\begin{array}{llll} 1 + x = 1 + \frac{n}{1}A & + \frac{n}{1}\cdot\frac{n-1}{2}A^2 & + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}A^3 & + \text{\&c.} \\ & \frac{n}{1}B & \frac{n}{1}\cdot\frac{n-1}{1}AB & \\ & & \frac{n}{1}C & \end{array}$$

Then equate the corresponding terms, and we have the first term $1 = 1$.

Again, the second term of the series $\frac{n}{1}A$, must be equal to the second term x of the binomial. For none of the other terms of the series are equipollent, or contain the same power of x, with the term $\frac{n}{1}A$. Not any of the terms A^2, A^3, A^4, &c; for they are double, triple, quadruple, &c, in power to A. Nor yet any of the terms containing B, C, D, &c; be-

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cause, by the supposition, they contain all different and increasing powers. It follows therefore, that $\frac{n}{1}$A makes up the whole value of the second term x of the given binomial. Consequently the second term A of the assumed series, contains only the first power of x; and the whole value of that term A is $= \frac{1}{n}x$.

But all the other equipollent terms of the expanded series must be equal to nothing, which is the general value of the terms, after the second, of the given quantity $1 + x$ or $1 + x + 0 + 0 + 0 +$ &c. Our business is therefore to find the several orders of equipollent terms of the expanded series. And these it is asserted will be as they are arranged above, in which B is equipollent with A^2, C with A^3, D with A^4, and so on.

Now that B is equipollent with A^2, is thus proved. The value of the third term is 0. But $\frac{n}{1} \cdot \frac{n-1}{2} A^2$ is a part of the third term. And it is only a part of that term: otherwise $\frac{n}{1} \cdot \frac{n-1}{2}$ would be $= 0$, which it is evident cannot happen in every value of n, as it ought; for indeed it happens only when n is $= 1$. Some other quantity then must be equipollent with $\frac{n}{1} \cdot \frac{n-1}{2} A^2$, and must be joined with it, to make up the whole third term equal to 0. Now that supplemental quantity can be no other than $\frac{n}{1}$B: for all the other following terms are evidently plupollent than B. It follows therefore, that B is equipollent with A^2, and contains the second power of x; or that $\frac{n}{1} \cdot \frac{n-1}{2} A^2 + \frac{n}{1} B = 0$, and consequently $\frac{n-1}{2} A^2 + B = 0$, or $B = \frac{1-n}{2} A^2 = \frac{1-n}{2n} Ax = \frac{1}{n} \cdot \frac{1-n}{2n} x^2$.

Again, the fourth term must be $= 0$. But the quantities $\frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} A^3 + \frac{n}{1} \cdot \frac{n-1}{2} AB$ are equipollent, and make up part of that fourth term. They are equipollent, or A^3 equipollent with AB, because A^2 and B are equipollent. And they do not constitute the whole of that term; for if they did, then would $\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}A^3 + \frac{n}{1}\cdot\frac{n-1}{2}AB$ be $=0$ in all values of n, or $\frac{n-2}{3n}A^2 + B = 0$: but it has been just shown above, that $\frac{n-1}{2}A^2 + B = 0$; it would therefore follow that $\frac{n-2}{3}$ would be $= \frac{n-1}{2}$, a circumstance which can only happen when $n = -1$, instead of taking place for every value of n. Some other quantity must therefore be joined with these to make up the whole of the fourth term. And this supplemental quantity can be no other than $\frac{n}{1}C$, because all the other following quantities are evidently plupollent than A^3 or AB. It follows therefore, that C is equipollent with A^3, and therefore contains the 3d power of x. And the whole value of C is

$$\frac{1-n}{2}\cdot\frac{n-2}{3}A^3+\frac{1-n}{1}AB=\frac{1-2n}{3}AB=\frac{1-2n}{3n}Bx=\frac{1}{n}\cdot\frac{1-n}{2n}\cdot\frac{1-2n}{3n}x^3.$$

And the process is the same for all the other following terms. Thus then we have proved the law of the whole series, both with respect to the coefficients of its terms, and to the powers of the letter x.

Since the above account was first written, almost 30 years ago, other demonstrations have been given by several ingenious and learned writers; which may be seen in some of the later volumes of the Philos. Trans. and elsewhere.

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