hence $1022 : 1015$, or $146 : 145 :: \frac{7}{4} : \frac{7}{4} \times \frac{145}{146} = (\frac{21}{4})^{\frac{1}{3}}$ nearly.

Consequently the square of this, or $(\frac{21}{4})^{\frac{2}{3}}$ will be $=$ $\frac{7^2}{4^2} \times \frac{145^2}{146^2} = \frac{1030225}{341056} = 3\frac{7057}{341056} = 3{\cdot}020690$, the quantity sought more nearly, being true in the last figure.

TRACT XI.

A NEW METHOD OF FINDING, IN FINITE AND GENERAL TERMS, NEAR VALUES OF THE ROOTS OF EQUATIONS OF THIS FORM, $x^n - px^{n-1} + qx^{n-2} - \&c = 0$; NAMELY, HAVING THE TERMS ALTERNATELY PLUS AND MINUS.

1. The following is one method more, to be added to the many we are already possessed of, for determining the roots of the higher equations. By means of it we readily find a root, which is sometimes accurate; and when not so, it is at least near the truth, and that by an easy finite formula, which is general for all equations of the above form, and of the same dimension, provided that root be a real one. This is of use for depressing the equation down to lower dimensions, and thence for finding all the roots, one after another, when the formula gives the root sufficiently exact; and when not, it serves as a ready means of obtaining a near value of a root, by which to commence an approximation still nearer, by the previously known methods of Newton, or Halley, or others. This method is further useful in elucidating the nature of equations, and certain properties of numbers; as will appear in some of the following articles. We have already easy methods for finding the roots of simple and quadratic equations,

I shall therefore begin with the cubic equation, and treat of each order of equations separately, in ascending gradually to the higher dimensions.

2. Let then the cubic equation $x^3 - px^2 + qx - r = 0$ be proposed. Assume the root $x = a$, either accurately or approximately, as it may happen, so that $x - a = 0$, accurately or nearly. Raise this $x - a = 0$ to the third power, the same dimension with the proposed equation,

so shall $x^3 - 3ax^2 + 3a^2x - a^3 = 0$;

but the proposed equation is $x^3 - \quad px^2 + \quad qx - r = 0$;

therefore the one of these is equal to the other. But the first term (x^3) of each is the same; and hence, if we assume the second terms equal between themselves, it will follow that the sum of the two remaining terms will also be equal, and give a simple equation by which the value of x is determined. Thus, $3ax^2$ being $= px^2$, or $a = \frac{1}{3}p$, and $3a^2x - a^3 = qx - r$, we hence have

$$x = \frac{a^3 - r}{3a^2 - q} = \frac{(\frac{1}{3}p)^3 - r}{3 \times (\frac{1}{3}p)^2 - q} = \frac{p^3 - 27r}{p^2 - 3q} \times \frac{1}{9}$$ by substituting $\frac{1}{3}p$, the value of a, instead of it.

3. Now this value of x here found, will be the middle root of the proposed cubic equation. For because a is assumed nearly or accurately equal to x, and also equal to $\frac{1}{3}p$, therefore x is $= \frac{1}{3}p$ nearly or accurately, that is, $\frac{1}{3}$ of the sum of the three roots, to which the coefficient p, of the second term of the equation, is always equal; and thus, being a medium among the three roots, it will be either nearly or accurately equal to the middle root of the proposed equation, when that root is a real one.

4. Now this value of x will always be the middle root *accurately*, whenever the three roots are in arithmetical progression; otherwise, only *approximately*. For when the three roots are in arithmetical progression, $\frac{1}{3}p$ or $\frac{1}{3}$ of their sum, it is well known, is equal to the middle term or root. In the other cases, therefore, the above-found value of x is only *near* the middle root.

5. When the roots are in arithmetical progression, because the middle term or root is then $=\frac{1}{3}p$, and also $=\frac{1}{9}\times\frac{p^3-27r}{p^2-3q}$, therefore $\frac{1}{3}p=\frac{1}{9}\times\frac{p^3-27r}{p^2-3q}$, or $2p^3=9pq-27r=9\times(pq-3r)$, an equation expressing the general relation of p, q, and r; where p is the sum of any three terms in arithmetical progression, q the sum of their three rectangles, and r the product of all the three. For, in any equation, the coefficient p of the second term, is the sum of the roots; the coefficient q of the third term, is the sum of the rectangles of the roots; and the coefficient r of the fourth term, is the sum of the solids of the roots, which in the case of the cubic equation is only one:—Thus, if the roots, or arithmetical terms, be 1, 2, 3. Here $p=1+2+3=6$, $q=1\times2+1\times3+2\times3=2+3+6=11$, $r=1\times2\times3=6$; then $2p^3=2\times6^3=432$, and $9\times(pq-3r)=9\times48=432$ also.

6. To illustrate now the rule $x=\frac{1}{9}\times\frac{p^3-27r}{p^2-3q}$ by some examples; let us in the first place take the equation $x^3-6x^2+11x-6=0$. Here $p=6$, $q=11$, and $r=6$; consequently $x=\frac{1}{9}\times\frac{p^3-27r}{p^2-3q}=\frac{1}{9}\times\frac{6^3-27\times6}{6^2-3\times11}=\frac{8-6}{12-11}=\frac{2}{1}=2$. This being substituted for x in the given equation, makes all the terms to vanish, and therefore it is an exact root, and the roots will be in arithmetical progression. Dividing therefore the given equation by $x-2=0$, the quotient is $x^2-4x+3=0$, the roots of which quadratic equation are 3 and 1, which are the other two roots of the proposed equation $x^3-6x^2+11x-6=0$.

7. If the equation be $x^3-39x^2+479x-1881=0$; we shall have $p=39$, $q=479$, and $r=1881$; then $x=\frac{1}{9}\times\frac{p^3-27r}{p^2-3q}=\frac{1}{9}\times\frac{39^3-27\times1881}{39^2-3\times479}=\frac{13^3-1881}{13^2-3\times479}=\frac{316}{28}=\frac{79}{7}=11\frac{2}{7}$. Then, substituting $11\frac{2}{7}$ for x in the proposed equation, the

negative terms are found to exceed the positive terms by 5, thus showing that $11\frac{2}{7}$ is very near, but something above, the middle root, and that therefore the roots are not in arithmetical progression. It is therefore probable that 11 may be the true value of the root, and on trial it is found to succeed. Then dividing $x^3 - 39x^2 + 479x - 1881$ by $x - 11$, the quotient is $x^2 - 28x + 171 = 0$, the roots of which quadratic equation are 9 and 19, the two other roots of the proposed equation.

8. If the equation be $x^3 - 6x^2 + 9x - 2 = 0$;
we shall have $p = 6$, $q = 9$, and $r = 2$; then $x =$

$$\frac{1}{9} \times \frac{p^3 - 27r}{p^2 - 3q} = \frac{1}{9} \times \frac{6^3 - 27 \times 2}{6^2 - 3 \times 9} = \frac{2^3 - 2}{12 - 9} = \frac{6}{3} = 2.$$

This value of x being substituted for it in the proposed equation, causes all the terms to vanish, as it ought, thus showing that 2 is the middle root, and that the roots are in arithmetical progression. Accordingly, dividing the given quantity $x^3 - 6x^2 + 9x - 2$ by $x - 2$, the quotient is $x^2 - 4x + 1 = 0$, a quadratic equation, whose roots are $2 + \sqrt{2}$ and $2 - \sqrt{2}$, the two other roots of the equation proposed.

9. If the equation be $x^3 - 5x^2 + 5x - 1 = 0$;
we shall have $p = 5$, $q = 5$, and $r = 1$; then $x =$

$$\frac{1}{9} \times \frac{5^3 - 27 \times 1}{5^2 - 3 \times 5} = \frac{1}{9} \times \frac{125 - 27}{25 - 15} = \frac{1}{9} \times \frac{98}{10} = \frac{49}{45} = 1\tfrac{4}{45}.$$

From which one might guess the root ought to be 1, and which being tried, is found to succeed. But without such trial, we might make use of this value $1\frac{4}{45}$, or $1\frac{1}{11}$ nearly, and approximate with it in the common way.

Having found the middle root to be 1, divide the given quantity $x^3 - 5x^2 + 5x - 1$ by $x - 1$, and the quotient is $x^2 - 4x + 1 = 0$, the roots of which are $2 + \sqrt{2}$, and $2 - \sqrt{2}$, the two other roots, as in the last article.

10. If the equation be $x^3 - 7x^2 + 18x - 18 = 0$;
we shall have $p = 7$, $q = 18$, and $r = 18$; then $x =$

$$\frac{1}{9} \times \frac{7^3 - 27 \times 18}{7^2 - 3 \times 18} = \frac{1}{9} \times \frac{343 - 486}{49 - 54} = \frac{143}{45} = 3\tfrac{8}{45} \text{ or } 3 \text{ nearly.}$$

Then trying 3 for x, it is found to succeed. And dividing $x^3-7x^2+18x-18$ by $x-3$, the quotient is $x^2-4x+6=0$, a quadratic equation whose roots are $2+\sqrt{-2}$ and $2-\sqrt{-2}$, the two other roots of the proposed equation, which are both impossible or imaginary.

11. If the equation be $x^3 - 6x^2 + 14x - 12 = 0$; we shall have $p = 6$, $q = 14$, and $r = 12$; then $x = \frac{1}{9} \times \frac{6^3-27\times12}{6^2-3\times14} = \frac{1}{9} \times \frac{216-324}{36-42} = \frac{108}{54} = 2$. Which being substituted for x, it is found to answer, the sum of the terms coming out $= 0$. Therefore the roots are in arithmetical progression. And, accordingly, by dividing $x^3 - 6x^2 + 14x - 12$ by $x - 2$, the quotient is $x^2 - 4x + 6 = 0$, the roots of which quadratic equation are $2+\sqrt{-2}$ and $2-\sqrt{-2}$, the two other roots of the proposed equation, and the common difference of the three roots is $\sqrt{-2}$.

12. But if the equation be $x^3 - 8x^2 + 22x - 24 = 0$; we shall have $p = 8$, $q = 22$, and $r = 24$; then $x = \frac{1}{9} \times \frac{8^3-27\times24}{8^2-3\times22} = \frac{1}{9} \times \frac{512-648}{64-66} = \frac{136}{18} = \frac{68}{9} = 7\frac{5}{9}$. Which being substituted for x in the proposed equation, the sum of the terms differs very widely from the truth, thereby showing that the middle root of the equation is an imaginary one, as it is indeed, the three roots being 4, and $2+\sqrt{-2}$, and $2-\sqrt{-2}$.

13. In Art. 2 the value of x was determined by assuming the second terms of the two equations, equal to each other. But a like near value might be determined by assuming either the two third terms, or the two fourth terms equal.

Thus the equations being $\begin{cases} x^3 - 3ax^2 + 3a^2x - a^3 = 0, \\ x^3 - px^2 + qx - r = 0, \end{cases}$

if we assume the third terms $3a^2x$ and qx equal, or $a=\sqrt{\frac{1}{3}q}$, the sums of the second and fourth terms will be equal, namely, $3ax^2 + a^3 = px^2 + r$; and hence we find

$$x = \sqrt{\frac{a^3-r}{p-3a}} = \sqrt{\frac{(\sqrt{\frac{1}{3}q})^3-r}{p-3\sqrt{\frac{1}{3}q}}},$$

by substituting $\sqrt{\frac{1}{3}q}$ the value of a instead of it.

And if we assume the fourth terms equal, namely $a^3 = r$, or $a = \sqrt[3]{r}$, then the sums of the second and third terms will be equal, namely, $3ax - 3a^2 = px - q$; and hence $x = \frac{q - 3a^2}{p - 3a} = \frac{q - 3r^{\frac{2}{3}}}{p - 3r^{\frac{1}{3}}}$, by substituting $r^{\frac{1}{3}}$ instead of a. And either of these two formulas will give nearly the same value of the root as the first formula, at least when the roots do not differ very greatly from one another.

But if they differ very much among themselves, the first formula will not be so accurate as these two others, because that in them the roots were more complexly mixed together; for the second formula is drawn from the coefficient of the third term, which is the sum of all the rectangles of the roots; and the third formula is drawn from the coefficient of the last term, which is equal to the continual product of all the roots; while the first formula is drawn from the coefficient of the second term, which is simply the sum of the roots. And indeed the last theorem is commonly the nearest of all. So that when we suspect the roots to be very wide of each other, it may be best to employ either the second or third formula.

Thus, in the equation $x^3 - 23x^2 + 62x - 40 = 0$, whose three roots are 1, 2, and 20. Here $p = 23$, $q = 62$, $r = 40$; and by the

$$\text{1st th. } x = \frac{1}{9} \times \frac{23^3 - 27 \times 40}{23^2 - 3 \times 62} = \frac{1}{9} \times \frac{12167 - 1080}{529 - 186} = 3\tfrac{3}{5} \text{ nearly,}$$

$$\text{2d th. } x = \sqrt{\frac{(\frac{62}{3})^{\frac{3}{2}} - 40}{23 - 3\sqrt{\frac{62}{3}}}} = \sqrt{\frac{94 - 40}{23 - 12{\cdot}87}} = \sqrt{5{\cdot}34} = 2\tfrac{1}{3} \text{ nearly.}$$

$$\text{3d th. } x = \frac{62 - 3 \times 40^{\frac{2}{3}}}{23 - 3 \times 40^{\frac{1}{3}}} = \frac{62 - 35{\cdot}1}{26 - 10\frac{1}{4}} = \frac{12}{7} = 1\tfrac{5}{7} \text{ nearly.}$$

Where the two latter are much nearer the middle root (2) than the first. And the mean between these two is $2\frac{1}{42}$, which is very near to that root. And this is commonly the case; the one being nearly as much too great as the other is too little.

14. To proceed now, in like manner, to the biquadratic equation, which is of this general form

$$x^4 - px^3 + qx^2 - rx + s = 0.$$

Assume the root $x = a$, or $x - a = 0$, and raise this equation $x - a = 0$ to the fourth power, or the same height with the proposed equation, which will give

$x^4 - 4ax^3 + 6a^2x^2 - 4a^3x + a^4 = 0$; but the proposed equation is $x^4 - px^3 + qx^2 - rx + s = 0$; therefore these two are equal to each other. Now if we assume the second terms equal, namely $4a = p$, or $a = \frac{1}{4}p$, then the sums of the three remaining terms will also be equal, namely,

$6a^2x^2 - 4a^3x + a^4 = qx^2 - rx + s$; and hence

$(6a^2 - q)\,x^2 - (4a^3 - r)\,x = s - a^4$, or

$(\frac{3}{8}p^2 - q)\,x^2 - (\frac{1}{16}p^3 - r)\,x = s - \frac{1}{256}p^4$ by substituting $\frac{1}{4}p$ instead of a: then, resolving this quadratic equation, we find its roots to be thus

$$x = \frac{p^3 - 16r \pm \sqrt{[(p^3 - 16r)^2 - (\frac{3}{2}p^2 - 4q) \times (p^4 - 256s)]}}{8 \times (\frac{3}{2}p^2 - 4q)};$$

or if we put $\text{A} = \frac{3}{2}p^2 - 4q$,

$\text{B} = p^3 - 16r$,

$\text{C} = p^4 - 256s$,

the two roots will be $x = \dfrac{\text{B} \pm \sqrt{(\text{B}^2 - \text{AC})}}{8\text{A}}$.

15. It is evident that the same property is to be understood here, as for the cubic equation in Art. 3, namely, that the two roots above found, are the middle roots of the four which belong to the biquadratic equation, when those roots are real ones; for otherwise the formulæ are of no use. But however those roots will not be accurate, when the sum of the two middle roots, of the proposed equation, is equal to the sum of the greatest and least roots, or when the four roots are in arithmetical progression; because that, in this case, $\frac{1}{4}p$, the assumed value of a, is neither of the middle roots exactly, but only a mean between them.

16. To exemplify this formula $x = \dfrac{\text{B} \pm \sqrt{(\text{B}^2 + \text{AC})}}{8\text{A}}$, let the proposed equation be $x^4 - 12x^3 + 49x^2 - 78x + 40 = 0$. Then

$$\text{A} = \tfrac{3}{2}p^2 - 4q = 12^2 \times \tfrac{3}{2} - 4 \times 49 = 216 - 196 = 20,$$
$$\text{B} = p^3 - 16r = 12^3 - 16 \times 78 = 1728 - 1248 = 480,$$
$$\text{C} = p^4 - 256s = 12^4 - 256 \times 40 = 20736 - 10240 = 10496.$$

Hence $x = \frac{\text{B} \pm \surd(\text{B}^2 - \text{AC})}{8\text{A}} = \frac{480 \pm \surd(480^2 - 20 \times 10496)}{8 \times 20} = \frac{15 \pm \surd 40}{5} = 3 \pm 1\frac{1}{4}$ nearly, or $4\frac{1}{4}$ and $1\frac{3}{4}$ nearly, or nearly 4 and 2, whose sum is 6. And trying 4 and 2, they are both found to answer, and therefore they are the two middle roots.

Then $(x - 4) \times (x - 2) = x^2 - 6x + 8$, by which dividing the given equation $x^4 - 12x^3 + 49x^2 - 78x + 40 = 0$, the quotient is $x^2 - 6x + 5 = 0$, the roots of which quadratic equation are 5 and 1, and which therefore are the greatest and least roots of the equation proposed.

17. If the equation be $x^4 - 12x^3 + 47x^2 - 72x + 36 = 0$; then

$$\text{A} = \tfrac{3}{2}p^2 - 4q = 12^2 \times \tfrac{3}{2} - 4 \times 47 = 216 - 188 = 28,$$
$$\text{B} = p^3 - 16r = 12^3 - 16 \times 72 = 1728 - 1152 = 576,$$
$$\text{C} = p^4 - 256s = 12^4 - 256 \times 36 = 20736 - 9216 = 11520.$$

Hence $x = \frac{\text{B} \pm \surd(\text{B}^2 - \text{AC})}{8\text{A}} = \frac{576 \pm \surd(576^2 - 28 \times 11520)}{8 \times 28} = \frac{18 \pm 3}{7} = 3$ and $2\frac{1}{7}$, or 3 and 2 nearly; both of which answer on trial; and therefore 3 and 2 are the two middle roots.

Then $(x - 3) \times (x - 2) = x^2 - 5x + 6 = 0$, by which dividing the given quantity $x^4 - 12x^3 + 47x^2 - 72x + 36 = 0$, the quotient is $x^2 - 7x + 6 = 0$, the roots of which quadratic equation are 6 and 1, which therefore are the greatest and least roots of the equation proposed.

18. If the equation be $x^4 - 7x^3 + 15x^2 - 11x + 3 = 0$; then

$$\text{A} = \tfrac{3}{2}p^2 - 4q = 7^2 \times \tfrac{3}{2} - 4 \times 15 = 73\tfrac{1}{2} - 60 = 13\tfrac{1}{2},$$
$$\text{B} = p^3 - 16r = 7^3 - 16 \times 11 = 343 - 176 = 167,$$
$$\text{C} = p^4 - 256s = 7^4 - 256 \times 3 = 2401 - 768 = 1633.$$

Hence $x = \frac{\text{B} \pm \surd(\text{B}^2 - \text{AC})}{8\text{A}} = \frac{167 \pm \surd(167^2 - 13\frac{1}{2} \times 1633)}{8 \times 13\frac{1}{2}} = \frac{167 \pm 76}{108} = 2\frac{1}{3}$ and $\frac{9}{10}$ nearly, or nearly 2 and 1; both which are found, on trial, to answer; and therefore 2 and 1 are the two middle roots sought.

Then $(x-2)\times(x-1)=x^2-3x+2$, by which dividing the given equation $x^4-7x^3+15x^2-11x+3=0$, the quotient is $x^2-4x+1=0$, the roots of which quadratic equation are $2+\sqrt{2}$ and $2-\sqrt{2}$, and which therefore are the greatest and least roots of the proposed equation.

19. But if the equa. be $x^4-9x^3+30x^2-46x+24=0$; then

$$\begin{aligned} \text{A} &= \tfrac{3}{2}p^2 - 4q = 9^2\times\tfrac{3}{2} - 4\times 30 = 121\tfrac{1}{2} - 120 = 1\tfrac{1}{2}, \\ \text{B} &= p^3 - 16r = 9^3 - 16\times 46 = 729 - 736 = -7, \\ \text{C} &= p^4 - 256s = 9^4 - 256\times 24 = 6561 - 6144 = 417. \end{aligned}$$

Hence $x=\dfrac{\text{B}\pm\sqrt{(\text{B}^2-\text{AC})}}{8\text{A}}=\dfrac{-7\pm\sqrt{(49-625\frac{1}{2})}}{8\times 1\frac{1}{2}}=$ $\dfrac{-7\pm\sqrt{-576\frac{1}{2}}}{12}$, an imaginary quantity, showing that the two middle roots are imaginary, and therefore the formula is of no use in this case, the four roots being $1, 2+\sqrt{-2}$, $2-\sqrt{-2}$, and 4.

20. And thus in other examples the two middle roots will be found when they are rational, or a near value when irrational, which in this case will serve for the foundation of a nearer approximation, to be made in the usual way.

We might also find another formula for the biquadratic equation, by assuming the last terms as equal to each other; for then the sum of the 2d, 3d, and 4th terms of each would be equal, and would form another quadratic equation, whose roots would be nearly the two middle roots of the biquadratic proposed.

21. Or a root of the biquadratic equation may easily be found, by assuming it equal to the product of two squares, as $(x-a)^2\times(x-b)^2=x^4-2(a+b)x^3+[2ab+(a+b)^2]x^2-2ab(a+b)x+a^2b^2=0$. For, comparing the terms of this with the terms of the equation proposed, in this manner, namely, making the second terms equal, then the third terms equal, and lastly the sums of the fourth and fifth terms equal, these equations will determine a near value of x by a simple equation. For those equations are

$p = 2(a + b)$, or $\frac{1}{2}p = a + b$,
$q = 2ab + (a + b)^2 = 2ab + \frac{1}{4}p^2$, or $2ab = q - \frac{1}{4}p^2$,
$rx - s = 2ab(a + b)x - a^2b^2 = \frac{1}{2}p(q - \frac{1}{4}p^2)x - \frac{1}{4}(q - \frac{1}{4}p^2)^2$;
Then the values of ab and $a + b$, found from the first and second of these equations, and substituted in the third, give $x = \frac{s - (\frac{1}{2}q - \frac{1}{8}p^2)^2}{r - p(\frac{1}{2}q - \frac{1}{8}p^2)} = \frac{64s - (4q - p^2)^2}{64r - 8p(4q - p^2)}$, a general formula for one of the roots of the biquadratic equation $x^4 - px^3 + qx^2 - rx + s = 0$.

22. To exemplify now this formula, let us take the same equation as in Art. 17, namely, $x^4 - 12x^3 + 47x^2 - 72x + 36 = 0$, the roots of which were there found to be 1, 2, 3, and 6. Then, by the last formula we shall have $x =$
$\frac{64s - (4q - p^2)^2}{64r - 8p(4q - p^2)} = \frac{64 \times 36 - (4 \times 47 - 12^2)^2}{64 \times 72 - 96(4 \times 47 - 12^2)} = \frac{64 \times 36 - 44 \times 44}{64 \times 72 - 96 \times 44}$
$= \frac{23}{24}$, or nearly 1, which is the least root.

23. Again, in the equation $x^4 - 7x^3 + 15x - 11x^2 + 3 = 0$, whose roots are 1, 2, $2 + \sqrt{2}$, and $2 - \sqrt{2}$, we have $x =$
$\frac{64 \times 3 - (60 - 49)^2}{64 \times 11 - 56(60 - 49)} = \frac{64 \times 3 - 11 \times 11}{64 \times 11 - 56 \times 11} = \frac{192 - 121}{704 - 616} = \frac{71}{88} = \frac{4}{5}$
nearly, which is nearly a mean between the two least roots 1 and $2 - \sqrt{2}$ or $\frac{3}{5}$ nearly.

24. But if the equation be $x^4 - 9x^3 + 30x^2 - 46x + 24 = 0$, which has impossible roots, the four roots being 1, $2 + \sqrt{-2}$, $2 - \sqrt{-2}$, and 4; we shall have $x =$
$\frac{64 \times 24 - (120 - 81)^2}{64 \times 46 - 72(120 - 81)} = \frac{64 \times 24 - 39 \times 39}{64 \times 46 - 72 \times 39} = \frac{1536 - 1521}{2944 - 2808} =$
$\frac{15}{136} = \frac{1}{9}$ nearly, which is of no use in this case of imaginary roots.

25. This formula will also sometimes fail when the roots are all real. As if the equation be $x^4 - 12x^3 + 49x^2 - 78x + 40 = 0$, the roots of which are 1, 2, 4, and 5. For here $x =$
$\frac{64 \times 40 - (196 - 144)^2}{64 \times 78 - 96(196 - 144)} = \frac{64 \times 40 - 52 \times 52}{64 \times 78 - 96 \times 52} = \frac{16 \times 10 - 13 \times 13}{16 \times 19\frac{1}{2} - 24 \times 13}$
$= \frac{160 - 169}{312 - 312} = \frac{-9}{0}$, which is of no use, being infinite.

26. For equations of higher dimensions, as the 5th, the 6th, the 7th, &c, we might, in imitation of this last method, combine other forms of quantities together. Thus, for the 5th power, we might compare it either with $(x-a)^4 \times (x-b)$, or with $(x-a)^3 \times (x-b)^2$, or with $(x-a)^3 \times (x-b) \times (x-c)$, or with $(x-a)^2 \times (x-b)^2 \times (x-c)$. And so for the other powers.

TRACT XII.

OF THE BINOMIAL THEOREM. WITH A DEMONSTRATION OF THE TRUTH OF IT IN THE GENERAL CASE OF FRACTIONAL EXPONENTS.

1. It is well known that this celebrated theorem is called *binomial*, because it contains a proposition of a quantity consisting of *two* terms, as a radix, to be expanded in a series of equal value. It is also called emphatically the Newtonian theorem, or Newton's binomial theorem, because he has commonly been reputed the author of it, as he was indeed for the case of fractional exponents, which is the most general of all, and includes all the other particular cases, of powers, or divisions, &c.

2. The binomial, as proposed in its general form, was, by Newton, thus expressed $\overline{P + PQ}^{\frac{m}{n}}$; where P is the first term of the binomial, Q the quotient of the second term divided by the first, and consequently PQ is the second term itself; or PQ may represent all the terms of a multinomial, after the first term, and consequently Q the quotient of all those terms, except the first term, divided by that first term, and may be either positive or negative; also $\frac{m}{n}$ represents the exponent of the binomial, and may denote any quantity, integral or