TRACT X.

THE INVESTIGATION OF CERTAIN EASY AND GENERAL RULES, FOR EXTRACTING ANY ROOT OF A GIVEN NUMBER.

1. THE roots of given numbers are commonly to be found, with much ease and expedition, by means of logarithms, when the indices of such roots are simple numbers, and the roots are not required to a great number of figures. And the square or cubic roots of numbers, to a good practical degree of accuracy, may be obtained, by inspection, by means of my tables of squares and cubes, published by order of the Commissioners of Longitude, in the year 1781. But when the indices of such roots are complex or irrational numbers; or when the roots are required to be found to a great many places of figures; it is necessary to make use of certain approximating rules, by means of the ordinary arithmetical computations. Such rules as are here alluded to, have only been discovered since the great improvements in the modern algebra: and the persons who have best succeeded in their enquiries after such rules, have been successively Sir Isaac Newton, Mr. Raphson, M. de Lagney, and Dr. Halley; who have shown, that the investigation of such theorems is also useful in discovering rules for approximating to the roots of all sorts of compound algebraical equations, to which the former rules, for the roots of all simple equations, bear a considerable affinity. It is presumed that the following short tract contains some advantages over any other method that has hitherto been given, both as to the ease and universality of the conclusions, and the general way in which the investigations are made: for here, a theorem is discovered, which, though it be general for all roots whatever, is at the same time

very accurate, and so simple and easy to use and to keep in mind, that nothing more so can be desired or hoped for; and further, that instead of searching out rules severally for each root, one after another, our investigation is at once for any indefinite possible root, by whatever quantity the index is expressed, whether fractional, or irrational, or simple, or compound.

2. In every theorem, or rule, here investigated,
N denotes the given number, whose root is sought,
n the index of that root,
a its nearest rational root, or a^n the nearest rational power to N, whether greater or less,
x the remaining part of the root sought, which may be either positive or negative, namely, positive when N is greater than a^n, otherwise negative. Hence then, the given number

N is $= (a+x)^n$, and the required root $N^{\frac{1}{n}} = a + x$.

3. Now, for the first rule, expand the quantity $(a+x)^n$ by the binomial theorem, so shall we have

$$N = (a+x)^n = a^n + na^{n-1}x + n \cdot \frac{n-1}{2} a^{n-2}x^2 + \&c.$$

Subtract a^n from both sides, so shall

$$N - a^n = n\,a^{n-1}x + n \cdot \frac{n-1}{2} a^{n-2}x^2 + \&c.$$

Divide by na^{n-1}, so shall

$$\frac{N-a^n}{n\,a^{n-1}} \text{ or } \frac{N-a^n}{n\,a^n} \times a = x + \frac{n-1}{2} \cdot \frac{x^2}{a} + \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{x^3}{a^2} + \&c.$$

Here, on account of the smallness of the quantity x in respect of a, all the terms of this series, after the first term, will be very small, and may therefore be neglected without much error, which gives $\frac{N-a^n}{n\,a^n}a$ for a near value of x, being only a small matter too great. And consequently

$a + x = \frac{N+(n-1)a^n}{n\,a^n}a$ is nearly $= N^{\frac{1}{n}}$ the root sought. And this may be accounted the first theorem.

4. Again, let the equation $N = a^n + na^{n-1}x + \&c$, be multiplied by $n-1$, and a^n added to each side, so shall we have

$(n-1)N + a^n = na^n + (n-1) . na^{n-1}x + \&c$, for a divisor: Also multiply the sides of the same equation by a and subtract a^{n+1} from each, so shall we have

$(N - a^n)a = na^n x + n . \frac{n-1}{2}a^{n-1}x^2 + \&c$, for a dividend:

Divide now this dividend by the divisor, so shall

$$\frac{N - a^n}{(n-1)N + a^n}a = x - \frac{n-1}{2} . \frac{x^2}{a} + \frac{n-1}{2} . \frac{n-2}{3} . \frac{x^3}{a} + \&c.$$

Which will be nearly equal to x, for the same reason as before; and this expression is about as much too little as the former expression was too great. Consequently, by adding a, we have $a + x$ or $N^{\frac{1}{n}}$ nearly $= \frac{nNa}{(n-1)N + a^n}$, for a second theorem, and which is nearly as much in defect as the former was in excess.

5. Now because the two foregoing theorems differ from the truth by nearly equal small quantities, if we add together the two numerators and the two denominators of the foregoing two fractional expressions, namely $\frac{N + (n-1)a^n}{na^n}a$ and $\frac{nN}{(n-1).N + a^n}a$, the sums will be the numerator and denominator of a new fraction, which will be much nearer than either of the former. The fraction so found is $\frac{\overline{n+1} . N + \overline{n-1} . a^n}{\overline{n-1} . N + \overline{n+1} . a^n}a$; which will be very nearly equal to $N^{\frac{1}{n}}$, or $a + x$, the root sought; for, by division, it is found to be equal to $a + x * - \frac{n-1}{2} . \frac{n+1}{6} . \frac{x^3}{a^2} + \&c$, where the term is wanting which contains the square of x, and the following terms are very small. And this is the third theorem.

6. A fourth theorem might be found by taking the arithmetical mean between the first and second, which would be

$(\frac{N+\overline{n-1}.a^n}{na^n}+\frac{nN}{\overline{n-1}.N+a^n})\times\frac{a}{2}$; which will be nearly of the same value, though not so simple, as the third theorem; for this arithmetical mean is found equal to

$$a + x * + \frac{n-1}{2}.\frac{n-2}{3}.\frac{x^3}{a^2} + \&c.$$

7. But the third theorem may be investigated in a more general way, thus: Assume a quantity of this form $\frac{pN+qa^n}{qN+pa^n}a$, with coefficients p and q to be determined from the process; the other letters N, a, n, representing the same things as before; then divide the numerator by the denominator, and make the quotient equal to $a+x$, so shall the comparison of the coefficients determine the relation between p and q required. Thus,

$$pN + qa^n = (p+q)a^n + pna^{n-1}x + pn.\frac{n-1}{2}a^{n-2}x^2 + \&c.$$

$$qN + pa^n = (p+q)a^n + qna^{n-1}x + qn.\frac{n-1}{2}a^{n-2}x^2 + \&c.$$

then dividing the former of these by the latter, we have

$$\frac{pN+qa^n}{qN+pa^n}a \text{ or } a+x = a+\frac{p-q}{p+q}nx+\frac{p-q}{p+q}n(\frac{n-1}{2}-\frac{qn}{p+q})\frac{x^2}{a}+\&c.$$

Then, by equating the corresponding terms, we obtain these three equations

$$a = a,$$

$$\frac{p-q}{p+q}n = 1,$$

$$\frac{n-1}{2} - \frac{qn}{p+q} = 0.$$

From which we find $\frac{p-q}{p+q} = \frac{1}{n}$ and $p : q :: n+1 : n-1$.

So that, by substituting $n+1$ and $n-1$, or any quantities proportional to them, for p and q, we shall have $\frac{\overline{n+1}.N+\overline{n-1}.a^n}{\overline{n-1}.N+\overline{n+1}.a^n}a$ for the value of the assumed quantity

$\frac{p\text{N}+qa^n}{q\text{N}+pa^n}a$, which is supposed nearly equal to $a + x$, the required root of the quantity N.

8. Now this third theorem $\frac{n+1 \cdot \text{N} + n-1 \cdot a^n}{n-1 \cdot \text{N} + n+1 \cdot a^n}a = \text{N}^{\frac{1}{n}}$,

which is general for roots, whatever be the value of n, and whether a^n be greater or less than N, includes all the rational formulas of De Lagney and Halley, which were separately investigated by them; and yet this general formula is perfectly simple and easy to apply, and easier kept in mind than any one of the said particular formulas. For, in words at length, it is simply this: to $n+1$ times N add $n-1$ times a^n, and to $n-1$ times N add $n+1$ times a^n, then the former sum multiplied by a and divided by the latter sum, will give the root $\text{N}^{\frac{1}{n}}$ nearly; or, as the latter sum is to the former sum, so is a, the assumed root, to the required root, nearly. Where it is to be observed that a^n may be taken either greater or less than N, but that the nearer it is to it, the better.

9. By substituting for n, in the general theorem, severally the numbers 2, 3, 4, 5, &c, we shall obtain the following particular theorems, as adapted to the 2d, 3d, 4th, 5th, &c, roots, namely, for the

2d or square root, $\frac{3\text{N} + a^2}{\text{N} + 3a^2}a$ - - - - $= \text{N}^{\frac{1}{2}}$

3d or cube root, $\frac{4\text{N} + 2a^3}{2\text{N} + 4a^3}a$, or $\frac{2\text{N} + a^3}{\text{N} + 2a^3}a = \text{N}^{\frac{1}{3}}$

4th root - - - $\frac{5\text{N} + 3a^4}{3\text{N} + 5a^4}a$ - - - - $= \text{N}^{\frac{1}{4}}$

5th root - - - $\frac{6\text{N} + 4a^5}{4\text{N} + 6a^5}a$, or $\frac{3\text{N} + 2a^5}{2\text{N} + 3a^5}a = \text{N}^{\frac{1}{5}}$

6th root - - - $\frac{7\text{N} + 5a^6}{5\text{N} + 7a^6}a$ - - - - $= \text{N}^{\frac{1}{6}}$

7th root - - - $\frac{8\text{N} + 6a^7}{6\text{N} + 8a^7}a$, or $\frac{4\text{N} + 3a^7}{3\text{N} + 4a^7}a = \text{N}^{\frac{1}{7}}$

&c.

10. To exemplify now our formula, let it be first required to extract the square root of 365. Here $N = 365$, $n = 2$, the nearest square is 361, whose root is 19.

$$\text{Hence } 3N + a^2 = 3 \times 365 + 361 = 1456,$$
$$\text{and } N + 3a^2 = 365 + 3 \times 361 = 1448;$$

then as $1448 : 1456 :: 19 : \frac{19 \times 182}{181} = 19\frac{19}{181} = 19\cdot10497$ &c.

Again, to approach still nearer, substitute this last found root $\frac{19 \times 182}{181}$ for a, the values of the other letters, remaining as before, we have $a^2 = \frac{19^2 \times 182^2}{181^2} = \frac{3458^2}{181^2}$; then

$$3N + a^2 = 3 \times 365 + \frac{3458^2}{181^2} = \frac{47831059}{32761},$$
$$N + 3a^2 = 365 + \frac{3 \times 3458^2}{181^2} = \frac{47831057}{32761}; \text{ hence}$$

$47831057 : 47831059 :: \frac{19 \times 182}{181}$ or $\frac{3458}{181} : \frac{3458 \times 47831059}{181 \times 47831057}$ = the root of 365 very exact, which being brought into decimals, would be true to about 20 places of figures.

11. For a second example, let it be proposed to double the cube, or to find the cube root of the number 2.

Here $N = 2$, $n = 3$, the nearest root $a = 1$, also $a^3 = 1$.

$$\text{Hence } 2N + a^3 = 4 + 1 = 5,$$
$$\text{and } N + 2a^3 = 2 + 2 = 4;$$

then as $4 : 5 :: 1 : \frac{5}{4} = 1\cdot25$ = the first approximation.

Again, take $a = \frac{5}{4}$, and consequently $a^3 = \frac{125}{64}$;

$$\text{Hence } 2N + a^3 = 4 + \frac{125}{64} = \frac{381}{64},$$
$$\text{and } N + 2a^3 = 2 + \frac{250}{64} = \frac{378}{64};$$

then $378 : 381$, or as $126 : 127 :: \frac{5}{4} : \frac{5}{4} \times \frac{127}{126} = \frac{635}{504} = 1\cdot259921$, for the cube root of 2, which is true in the last figure.

And by taking $\frac{635}{504}$ for the value of a, and repeating the process, a great many more figures may be found.

12. For a third example let it be required to find the 5th root of 2.

Here $N = 2$, $n = 5$, the nearest root $a = 1$.

Hence $3N + 2a^5 = 6 + 2 = 8$,

and $2N + 3a^5 = 4 + 3 = 7$;

then as $7 : 8 :: 1 : \frac{8}{7} = 1\frac{1}{7}$ for the first approximation.

Again, taking $a = \frac{8}{7}$, we have

$$3N + 2a^5 = 6 + \frac{65536}{16807} = \frac{166378}{16807},$$

$$2N + 3a^5 = 4 + \frac{98304}{16807} = \frac{165532}{16807};$$

then $165532 : 166378 :: \frac{8}{7} : \frac{8}{7} \times \frac{83189}{82766} = \frac{4}{7} \times \frac{83189}{41383} = \frac{332756}{289681}$ $= 1{\cdot}148698$ &c, for the 5th root of 2, true in the last figure.

13. To find the 7th root of $126\frac{1}{5}$.

Here $N = 126\frac{1}{5}$, $n = 7$, the nearest root $a = 2$, also $a^7 = 128$.

Hence $4N + 3a^7 = 504\frac{4}{5} + 384 = 888\frac{4}{5} = \frac{4444}{5}$,

and $3N + 4a^7 = 378\frac{3}{5} + 512 = 890\frac{3}{5} = \frac{4453}{5}$;

then $4453 : 4444 :: 2 : \frac{8888}{4453} = 1{\cdot}995957$, root very exact by one operation, being true to the nearest unit in the last figure.

14. To find the 365th root of $1{\cdot}05$, or the amount of 1 pound for 1 day, at 5 per cent. per annum, compound interest.

Here $N = 1{\cdot}05$, $n = 365$, $a = 1$ the nearest root.

Hence $366N + 364a = 748{\cdot}3$,

and $364N + 366a = 748{\cdot}2$;

then as 748·2 : 748·3 : : 1 : $\frac{7483}{7482} = 1\frac{1}{7482} = 1\cdot00013366$, the root sought, very exact at one operation.

15. Required to find the value of the quantity $(5\frac{1}{4})^{\frac{2}{3}}$ or $(\frac{21}{4})^{\frac{2}{3}}$. Now this may be done two ways; either by finding the $\frac{2}{3}$ power or $\frac{3}{2}$ root of $\frac{21}{4}$ at once; or else by finding the 3d or cubic root of $\frac{21}{4}$, and then squaring the result.

By the first way:—Here it is easy to see that a is nearly $= 3$, because $3^{\frac{3}{2}} = \sqrt{27} = 5 +$ some small fraction. Hence, to find nearly the square root of 27, or $\sqrt{27}$, the nearest power to which is $25 = a^2$ in this case:

$$\text{Hence } 3\text{N} + a^2 = 3 \times 27 + 25 = 106,$$
$$\text{and } \text{N} + 3a^2 = 27 + 3 \times 25 = 102;$$

then 102 : 106, or 51 : 53 : : 5 : $\frac{5 \times 53}{51} = \frac{265}{51} = \sqrt{27}$ nearly.

Then having $\text{N} = \frac{21}{4}$, $n = \frac{3}{2}$, $a = 3$, and $a^{\frac{3}{2}} = \frac{265}{51}$ nearly;

it will be $\frac{5}{2}\text{N} + \frac{1}{2} a^{\frac{3}{2}} = \frac{5}{2} \times \frac{21}{4} + \frac{1}{2} \times \frac{265}{51} = \frac{6415}{408}$,

and $\frac{1}{2}\text{N} + \frac{5}{2} a^{\frac{3}{2}} = \frac{1}{2} \times \frac{21}{4} + \frac{5}{2} \times \frac{265}{51} = \frac{6371}{408}$,

hence 6371 : 6415 : : 3 : $\frac{19245}{6371} = 3\frac{134}{6371} = 3\cdot020719$, for the value of the quantity sought nearly, by this way.

Again, by the other method, in finding first the value of $(\frac{21}{4})^{\frac{1}{3}}$, or the cube root of $\frac{21}{4}$. It is evident that 2 is the nearest integer root, being the cube root of $8 = a^3$.

$$\text{Hence } 2\text{N} + a^3 = \tfrac{21}{2} + 8 = \tfrac{74}{4},$$
$$\text{and } \text{N} + 2a^3 = \tfrac{21}{4} + 16 = \tfrac{85}{4};$$

then 85 : 74 : : 2 : $\frac{148}{85}$, or $= \frac{7}{4}$ nearly. Then taking $\frac{7}{4}$ for a,

$$\text{we have } 2\text{N} + a^3 = \frac{21}{2} + \frac{343}{64} = \frac{1015}{64},$$

$$\text{and } \text{N} + 2a^3 = \frac{21}{4} + \frac{2.343}{64} = \frac{1022}{64};$$

hence $1022 : 1015$, or $146 : 145 :: \frac{7}{4} : \frac{7}{4} \times \frac{145}{146} = (\frac{21}{4})^{\frac{1}{3}}$ nearly.

Consequently the square of this, or $(\frac{21}{4})^{\frac{2}{3}}$ will be $=$
$\frac{7^2}{4^2} \times \frac{145^2}{146^2} = \frac{1030225}{341056} = 3\frac{7057}{341056} = 3{\cdot}020690$, the quantity sought more nearly, being true in the last figure.

TRACT XI.

A NEW METHOD OF FINDING, IN FINITE AND GENERAL TERMS, NEAR VALUES OF THE ROOTS OF EQUATIONS OF THIS FORM, $x^n - px^{n-1} + qx^{n-2} - \&c = 0$; NAMELY, HAVING THE TERMS ALTERNATELY PLUS AND MINUS.

1. The following is one method more, to be added to the many we are already possessed of, for determining the roots of the higher equations. By means of it we readily find a root, which is sometimes accurate; and when not so, it is at least near the truth, and that by an easy finite formula, which is general for all equations of the above form, and of the same dimension, provided that root be a real one. This is of use for depressing the equation down to lower dimensions, and thence for finding all the roots, one after another, when the formula gives the root sufficiently exact; and when not, it serves as a ready means of obtaining a near value of a root, by which to commence an approximation still nearer, by the previously known methods of Newton, or Halley, or others. This method is further useful in elucidating the nature of equations, and certain properties of numbers; as will appear in some of the following articles. We have already easy methods for finding the roots of simple and quadratic equations,