TRACT IX.

A METHOD OF SUMMING THE SERIES $a + bx + cx^2 + dx^3 + ex^4 + \&c$, WHEN IT CONVERGES VERY SLOWLY, NAMELY, WHEN x IS NEARLY EQUAL TO 1, AND THE COEFFICIENTS $a, b, c, d, \&c$, DECREASE VERY SLOWLY: THE SIGNS OF ALL THE TERMS BEING POSITIVE.

ARTICLE 1.

WHEN there is occasion to find the sum of such series as that above-mentioned, having the coefficients a, b, c, d, &c, of the terms, decreasing very slowly, and the converging quantity x pretty large; the sum cannot be found by collecting the terms together, on account of the immense number of them which it would be necessary to collect; neither can it be summed by means of the differential series, because the powers of the quantity $\frac{x}{1-x}$ will then diverge faster than the differential coefficients converge. In such case then we must have recourse to some other method of transforming it into another series which shall converge faster. The following is a method by which the proposed series is changed into another, which converges so much the quicker as the original series is slower.

2. The method is thus. Assume $\frac{a^2}{D}$ = the given series

$$a + bx + cx^2 + dx^3 + \&c. \quad \text{Then shall}$$

D be $= \frac{a^2}{a+bx+cx^2+\&c}$; which, by actual division, is $= a - bx - (c - \frac{b^2}{a})x^2 - (d - \frac{2bc}{a} + \frac{b^3}{a^2})x^3 - (e - \frac{2bd + c^2}{a} + \frac{3b^2c}{a^2} - \frac{b^4}{a^3})\,x^4 -$ &c. Consequently a^2 divided by this series will be equal to the series proposed; and this new series will be very easily summed, in comparison with the original one, because all the coefficients after the second term are evidently very small; and indeed they are so much the smaller, and fitter for summation, by how much the coefficients of the original series are nearer to equality; so that, when these a, b, c, d, &c, are quite equal, then the third, fourth, &c, coefficients of the new series become equal to nothing, and the sum accurately equal to $\frac{a^2}{a-bx} = \frac{a^2}{a-ax} = \frac{a}{1-x}$; which is also known to be true from other principles.

3. Though the first two terms, $a - bx$, of the new series, be very great in comparison with each of the following terms, yet these latter may not always be small enough to be entirely rejected when much accuracy is required in the summation. And in such case it will be necessary to collect a great number of them, to obtain their sum pretty near the truth; because their rate of converging is but small, it being indeed pretty much like to the rate of the original series, but only the terms, each to each, are much smaller, and that commonly in a degree to the hundredth or thousandth part.

4. The resemblance of this new series however, beginning with the third term, to the original one, in the law of progression, intimates to us that it will be best to sum it in the very same manner as the former. Hence then putting

$$a' = c - \frac{b^2}{a},$$

$$b' = d - \frac{2bc}{a} + \frac{b^3}{a^2},$$

$$c' = e - \frac{2bd + c^2}{a} + \frac{3b^2c}{a^2} - \frac{b^4}{a^3},$$

&c,

and consequently the proposed series $a + bx + cx^2 +$ &c,

$$= \frac{a^2}{a - bx - a'x^2 - b'x^3 - c'x^4 \&c} = \frac{a^2}{a - bx - x^2 \times (a' + b'x + c'x^2 \&c)},$$

by taking the sum of the series $a' + b'x + c'x^2 +$ &c, by the very same theorem as before, the sum s of the original series will then be expressed thus, s =

$$\cfrac{a^2}{a - bx - \cfrac{a'^2 x^2}{a' - b'x - (c' - \frac{b'^2}{a'})x^2 - (d' - \frac{2b'c'}{a'} + \frac{b'^3}{a'^2})x^3 - \&c;}};$$

where the series in the last denominator, having again the same properties with the former one, will have its first two terms very large in respect of the following terms. But these first two terms, $a' - b'x$, are themselves very small in comparison with the first two, $a - bx$, of the former series; and therefore much more are the third, fourth, &c, terms of this last denominator, very small in comparison with the same $a - bx$: for which reason they may now perhaps be small enough to be neglected.

5. But if these last terms be still thought too large to be omitted, then find the sum of them by the very same theorem: and thus proceed, by repeating the operation in the same manner, till the required degree of accuracy is obtained. Which it is evident, will happen after a small number of repetitions, because that, in each new denominator, the third, fourth, &c, terms, are commonly depressed, in the scale of numbers, two or three places lower than the first and second terms are. And the general theorem, denoting the sum s when the process is continually repeated, will be this,

$$\cfrac{aa}{a - bx - \cfrac{a'a'xx}{a' - b'x - \cfrac{a''a''xx}{a'' - b''x - \cfrac{a'''a'''xx}{a''' - b'''x - \cfrac{a^{iv}a^{iv}xx}{a'''' - b''''x \ \&c.}}}}}$$

6. But the general denominator D in the fraction $\frac{a^2}{D}$, which is assumed for the value of s or $a + bx + cx^2 + \&c$, may be otherwise found as below; from which the general law of

the values of the coefficients will be rendered visible. Assume s or $a + bx + cx^2 + \&c$,

$$\text{or } \frac{a^2}{D} = \frac{a^2}{a - bx - a'x^2 - b'x^3 - c'x^4 - \&c}\text{; then shall}$$

$$a^2 = a + bx + cx^2 + \&c \quad \times a - bx - a'x^2 - b'x^3 - \&c$$

$$\begin{array}{rllllll} = a^2 + abx & + acx^2 & + adx^3 & + aex^4 & + afx^5 & + \&c \\ -ab & - bb & - bc & - bd & - be & \\ & - a'a & - a'b & - d'c & - a'd & \\ & & - b'a & - b'b & - b'c & \\ & & & - c'a & - c'b & \\ & & & & - d'a. & \end{array}$$

Hence, by equating the coefficients of the like terms to nothing, we obtain the following general values:

$$a' = c - \frac{bb}{a},$$

$$b' = d - \frac{ba' + cb}{a},$$

$$c' = e - \frac{bb' + ca' + db}{a},$$

$$d' = f - \frac{bc' + cb' + da' + eb}{a},$$

$$e' = g - \frac{bd' + cc' + db' + ea' + fb}{a},$$

&c.

Where the values of the coefficients are the same in effect as before found, but here the law of their continuation is manifest.

7. To exemplify now the use of this method, let it be proposed to sum the very slow series $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \&c.$ when $x = \frac{9}{10} = \cdot 9$, denoting the hyp. log. of $\frac{1}{1-x}$, or, in this case, of 10.

Now it will be proper, in the first place, to collect a few of the first terms together, and then apply the theorem to the remaining terms, which, by being nearer to an equality than the terms are near the beginning of the series, will be fitter to receive the application of the theorem. Thus to collect the first 12 terms:

No.	Powers of x.	The first 12 terms, found by dividing x, x^2, x^3, &c, by the numbers 1, 2, 3, &c,
1	·9 - - - - -	·9
2	·81 - - - -	·405
3	·729 - - - -	·243
4	·6561 - - -	·164025
5	·59049 - - -	·118098
6	·531441 - -	·0885735
7	·4782969 - -	·06832812857
8	·43046721 - -	·05380840125
9	·387420489 -	·043046721
10	·3486784401 -	·03486784401
11	·31381059609 -	·02852823601
12	·282429536481	·02353579471
13	·2541865828329	2·17081162555 the sum of 12 terms.

Then we have to find the sum of the rest of the terms after these first 12, namely of $x^{13} \times (\frac{1}{13} + \frac{1}{14}x + \frac{1}{15}x^2 + \frac{1}{16}x^3 + \&c)$, in which $x = {\cdot}9$, and $x^{13} = {\cdot}2541865828329$; also $a = \frac{1}{13}$, $b = \frac{1}{14}$, $c = \frac{1}{15}$, &c, and the first application of our rule, gives, for the value of $\frac{1}{13} + \frac{1}{14}x + \frac{1}{15}x^2 + \&c$, or s,

$$\frac{(\frac{1}{13})^2 = {\cdot}005917159763 \ \&c}{{\cdot}012637363 - x^2 \times : {\cdot}000340136 + {\cdot}000279397x + {\cdot}000233592x^2 + \&c};$$

the second gives

$$\frac{{\cdot}00591715976}{{\cdot}012637363 - \dfrac{{\cdot}000340136^2 x^2}{{\cdot}000088678 - x^2 \times : {\cdot}000004086 + {\cdot}000003060x + \&c}};$$

the third gives

$$\frac{{\cdot}00591715976}{{\cdot}012637363 - \dfrac{{\cdot}000340136^2 x^2}{{\cdot}000088678 - \dfrac{{\cdot}000004087^2 x^2}{{\cdot}000001333 - x^2 \times : {\cdot}000000089 + \&c}}};$$

the fourth gives

$$\frac{{\cdot}00591715976}{{\cdot}012637363 - \dfrac{{\cdot}000340136^2 x^2}{{\cdot}000088678 - \dfrac{000004087^2 x^2}{{\cdot}000001333 - \dfrac{{\cdot}000000089^2 x^2}{{\cdot}0000000344}}}}.$$

Or, when the terms in the numerators are squared, it is

$$\cfrac{\cdot 00591715976}{\cdot 012637363-\cfrac{\cdot 000000093710985}{\cdot 000088678-\cfrac{\cdot 000000000013526212}{\cdot 000001333-\cfrac{\cdot 000000000000006416}{\cdot 0000000344}}}}$$

Or, by omitting a proper number of ciphers, it is

$$\cfrac{\cdot 0591715976}{\cdot 12637363-\cfrac{\cdot 0093710985}{\cdot 88678-\cfrac{\cdot 013526212}{\cdot 1333-\cfrac{\cdot 006416}{\cdot 344-z}}}}$$

An unknown quantity z is here placed after the last denominator, to represent the small quantity to be subtracted from the said denominator 344. Now, rejecting the small quantity z, and beginning at the last fraction to calculate, their values will be as here ranged in the first annexed column.

Fractions.	1. Ra.	2. Ra.	3. Ratio.	4. Ratio.
·518200000	425			
1218931	106	4·01	2·39	
11799	63	1·68	$\frac{1\cdot 68\times 187}{63z}$	$\frac{2\cdot 39\times 63z}{1\cdot 68\times 187}$
187	$\frac{187}{z}$	$\frac{63z}{187}$		
z				
$4\frac{3}{10}$	44	1·43	1·18	2·03

placing z below them for the next unknown fraction. Divide then every fraction by the next below it, placing the quotients or ratios in the next column. Then take the quotients or ratios of these; and so on till the last ratio $\frac{2\cdot 39\times 63z}{1\cdot 68\times 187}$, which, from the nature of the series of the first terms of every column, must be less than the next preceding one 2·39: consequently z must be less than $\frac{1\cdot 68\times 187}{63}$, or less than 5. But, from the nature of the series in the vertical row, or column of first ratios, $\frac{187}{z}$ must be less than 63; and consequently z must be greater than $\frac{187}{63}$, or greater than 3. Since then

z is less than 5 and greater than 3, it is probable that the mean value 4 is near the truth: and accordingly taking 4 for z, or rather 4·3, as z appears to be nearer 5 than 3, and taking the continual ratios, as placed along the last line of the table, their values are found to accord very well with the next preceding numbers, both in the columns and oblique rows.

Hence, using ·043 for z in the denominator $\cdot 344 - z$ of the last fraction of the general expression, and computing from the bottom, upwards through the whole, the quotients, or values of the fractions, in the inverted order, will be

213
12079
1223397
·518414000

of which the last must be nearly the value of the series $\frac{1}{13} + \frac{1}{14}x + \frac{1}{15}x^2 +$ &c, when $x = \cdot 9$.

Then this value ·518414 of the series, being multiplied by x^{13} or ·2541865828329, gives ·1317738 for the sum of all the terms of the original series after the first 12 terms; to which therefore the sum of the first 12 terms, or 2·17081162, being added, we have 2·30258542 for the sum of the original series $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 +$ &c. Which value is true within about 3 in the 8th place of figures, the more accurate value being 2·30258509 &c, or the hyp. log. of 10.

N. B. By prop. 8 Stirling's Summat.; and by cor. 4, p. 65 Simpson's Dissert. the series $a + bx + cx^2 + dx^3 +$ &c, transforms to

$$\frac{1}{1-x} \times [a - \text{D}(\frac{x}{1-x}) + \text{D}'(\frac{x}{1-x})^2 - \text{D}''(\frac{x}{1-x})^3 + \text{D}'''(\frac{x}{1-x})^4 -].$$

And thus the series $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 +$ &c, becomes

$$\frac{x}{1-x} \times [1 - \tfrac{1}{2}(\frac{x}{1-x}) + \tfrac{1}{3}(\frac{x}{1-x})^2 - \tfrac{1}{4}(\frac{x}{1-x})^3 + \text{\&c}],$$ which may be summed by our method.