TRACT VIII.

A NEW METHOD FOR THE VALUATION OF NUMERAL INFINITE SERIES, WHOSE TERMS ARE ALTERNATELY (+) PLUS AND (−) MINUS; BY TAKING CONTINUAL ARITHMETICAL MEANS BETWEEN THE SUCCESSIVE SUMS, AND THEIR MEANS.

ARTICLE 1.

The remarkable difference between the facility which mathematicians have found, in their endeavours to determine the values of infinite series, whose terms are alternately affirmative and negative, and the difficulty of doing the same thing with respect to those series whose terms are all affirmative, is one of those striking circumstances in science which we can hardly persuade ourselves is true, even after we have seen many proofs of it; and which serve to put us ever after on our guard not to trust to our first notions, or conjectures, on these subjects, till we have brought them to the test of demonstration. For, at first sight it is very natural to imagine, that those infinite series whose terms are all affirmative, or added to the first term, must be much simpler in their nature, and much easier to be summed, than those whose terms are alternately affirmative and negative; which, however, we find, on examination, to be directly the reverse; the methods of finding the sums of the latter series being numerous and easy, and also very general, whereas those that have been hitherto discovered for the summation of the former series, are few and difficult, and confined to series whose terms are generated from each other according to some particular laws, instead of extending, as the other methods do, to all sorts of series, whose terms are connected together by addition, by whatever law their terms are formed. Of this remarkable difference between these two sorts of series, the new method of finding the sums of those whose terms are alternately positive and negative, which is the subject of the present tract, will afford us a striking instance, as it possesses the happy qualities of simplicity, ease, perspicuity, and universality; and yet, as the essence of it consists in the alternation of the signs + and —, by which the terms are connected with the first term, it is of no use in the summation of those other series whose terms are all connected with each other by the sign +.

2. This method, so easy and general, is, in short, simply this: beginning at the first term a of the series $a - b + c - d + e - f +$ &c, which is to be summed, compute several successive values of it, by taking in successively more and more terms, one term being taken in at a time; so that the first value of the series shall be its first term a, or even 0 or nothing may begin the series of sums; the next value shall be its first two terms $a - b$, reduced to one number; its next value shall be the first three terms $a - b + c$, reduced to one number; its next value shall be the first four terms $a - b + c - d$, reduced also to one number; and so on. This, it is evident, may be done by means of the easy arithmetical operations of addition and subtraction. And then, having found a sufficient number of successive values of the series, more or less as the case may require, interpose between these values a set of arithmetical mean quantities or proportionals; and between these arithmetical means interpose a second set of arithmetical mean quantities; and between these arithmetical means of the second set, interpose a third set of arithmetical mean quantities; and so on as far as you please. By this process we soon find either the true value of the series proposed, when it has a determinate rational value, or otherwise we obtain several sets of values approximating nearer and nearer to the sum of the series, both in the columns and in the lines, either horizontal or obliquely de-

scending or ascending; namely, both of the several sets of means themselves, and the sets or series formed of any of their corresponding terms, as of all their first terms, of their second terms, of their third terms, &c, or of their last terms, of their penultimate terms, of their antepenultimate terms, &c: and if between any of these latter sets, consisting of the like or corresponding terms of the former sets of arithmetical means, we again interpose new sets of arithmetical means, as we did at first with the successive sums, we shall obtain other sets of approximating terms, having the same properties as the former. And thus we may repeat the process as often as we please, which will be found very useful in the more difficult diverging series, as we shall see hereafter. For this method, being derived only from the circumstance of the alternation of the signs of the terms, + and —, it is therefore not confined to converging series alone, but is equally applicable both to diverging series, and to *neutral* series, by which last name I shall take the liberty to distinguish those series, whose terms are all of the same constant magnitude; namely, the application is equally the same for all the three following sorts of series, viz.

Converging, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} +$ &c.
Diverging, $1 - 2 + 3 - 4 + 5 - 6 +$ &c.
Neutral, $1 - 1 + 1 - 1 + 1 - 1 +$ &c.

As is demonstrated in what follows, and exemplified in a variety of instances.

It must be noted, however, that by the value of the series, I always mean such *radix*, or finite expression, as, by evolution, would produce the series in question; according to the sense we have stated in the former paper, on this subject; or an approximate value of such radix; and which radix, as it may be substituted instead of the series in any operation, I call the value of the series.

3. It is an obvious and well-known property of infinite series, with alternate signs, that when we seek their value by collecting their terms one after another, we obtain a series of successive sums, which approach continually nearer and

nearer to the true value of the proposed series, when it is a converging one, or one whose terms always decrease by some regular law; but in a diverging series, or one whose terms as continually increase, those successive sums diverge always more and more from the true value of the series. And from the circumstance of the alternate change of the signs, it is also a property of those successive sums, that when the last term which is included in the collection, is a positive one, then the sum obtained is too great, or exceeds the truth; but when the last collected term is negative, then the sum is too little, or below the truth. So that, in both the converging and diverging series, the first term alone, being positive, exceeds the truth; the second sum, or the sum of the first two terms, is below the truth; the third sum, or the sum of the three terms, is above the truth; the fourth sum, or the sum of four terms, is below the truth; and so on; the sum of any even number of terms being below the true value of the series, and the sum of any odd number, above it. All which is generally known, and evident from the nature and form of the series. So, of the series $a - b + c - d + e - f +$ &c, the first sum a is too great; the second sum $a - b$ too little; the third sum $a - b + c$ too great; and so on as in the following table, where s is the true value of the series, and 0 is placed before the collected sums, to complete the series, being the value when no terms are included:

	Successive sums.
s is greater than	0
s is less than	a
s is greater than	$a - b$
s is less than	$a - b + c$
s is greater than	$a - b + c - d$
s is less than	$a - b + c - d + e$
&c.	&c.

4. Hence the value of every alternate series s, is positive, and less than the first term a, the series being always supposed to begin with a positive term a; and consequently, if the signs of all the terms be changed, or if the series begin

with a negative term, the value s will still be the same, but negative, or the sign of the sum will be changed, and the value become $-s = -a + b - c + d -$ &c. Also, because the successive sums, in a converging series, always approach nearer and nearer to the true value, while they recede always farther and farther from it in a diverging one; it follows that, in a neutral series, $a - a + a - a +$ &c, which holds a middle place between the two former, the successive sums $0, a, 0, a, 0, a,$ &c, will neither converge nor diverge, but will be always at the same distance from the value of the proposed series $a - a + a - a +$ &c, and consequently that value will always be $= \frac{1}{2}a$, which holds every where the middle place between 0 and a.

I am not unaware that, though $a - a + a - a + -$ &c, may be produced by evolving $\frac{a^2}{a + a}$ by actual division, it will also arise by evolving several other functions in like manner; as

$$\frac{a^2}{a + a + a}, \text{ or } \frac{a^2}{a + a + a + a}, \text{ \&c, or } \frac{a^2 + a^2 + a^2 + \text{\&c}}{a + a + a + a + \text{\&c}},$$

or any other similar function, in which the numerator has fewer terms than the denominator. Yet the preference among them all seems justly due to the first

$\frac{a^2}{a + a} = \frac{a^2}{2a} = \frac{a}{2} = \frac{1}{2}a$, for this reason, besides what is said above, viz, put s for the value of the series $a - a + a - a +$ &c: since

then $s = a - a + a - a + -$ &c,
and $a = a$, take the upper equ. from the under,
then $a - s = a - a + a - a +$ &c $= s$ by sup.
theref. $a - s = s$, and $2s = a$, or $s = \frac{1}{2}a$, as above.

5. Now, with respect to a converging series, $a - b + c - d +$ &c; because 0 is below, and a above s, the value of the series, but a nearer than 0 to the value s, it follows that s lies between a and $\frac{1}{2}a$, and that $\frac{1}{2}a$ is less than s, and so nearer to s than 0 is. In like manner, because a is above, and $a - b$ below the value s, but $a - b$ nearer to that value than a is,

it follows that s lies between a and $a - b$, and that the arithmetical mean $a - \frac{1}{2}b$ is something above the value of s, but nearer to that value than a is. And thus, the same reasoning holding in every following pair of successive sums, the arithmetical means between them will form another series of terms, which are, like those sums, alternately less and greater than the value of the proposed series, but approximating nearer to that value than the several successive sums do, as every term of those means is nearer to the value s, than the corresponding preceding term in the sums is. And, like as the successive sums form a progression approaching always nearer and nearer to the value of the series; so, in like manner, their arithmetical means form another progression, coming nearer and nearer to the same value, and each term of the progression of means nearer than each term of the successive sums. Hence then we have the two following series, namely, of successive sums and their arithmetical means, in which each step approaches nearer to the value of s than the former, the latter progression being however nearer than the former, and the terms or steps of each alternately below and above the value s of the series $a - b + c - d +$ &c.

Successive sums.	Arithmetical means.
⊐ 0	⊐ $\frac{1}{2}a$
⊏ a	⊏ $a - \frac{1}{2}b$
⊐ $a - b$	⊐ $a - b + \frac{1}{2}c$
⊏ $a - b + c$	⊏ $a - b + c - \frac{1}{2}d$
⊐ $a - b + c - d$	⊐ $a - b + c - d + \frac{1}{2}e$
⊏ $a - b + c - d + e$	⊏ $a - b + c - d + e - \frac{1}{2}f$
&c.	&c.

where the mark ⊐, placed before any step, signifies that it is too little, or below the value s of the converging series $a - b + c - d +$ &c; and the mark ⊏ signifies the contrary, or too great. And hence $\frac{1}{2}a$, or half the first term of such a converging series, is less than s the value of the series.

6. And since these two progressions possess the same properties, but only the terms of the latter nearer to the truth than the former; for the very same reasons as before, the means between the terms of these first arithmetical means, will form a third progression, whose terms will approach still nearer to the value of s than the second progression, or the first means; and the means of these second means will approach nearer than the said second means do; and so on continually, every succeeding order of arithmetical means, approaching nearer to the value of s than the former. So that the following columns of sums and means will be each nearer to the value of s than the former, viz.

	Suc. sums.	1st means.	2d means.	3d means.
⊐	0	$\frac{a}{2}$	$\frac{3a-b}{4}$	$\frac{7a-4b+c}{8}$
⊏	a	$a-\frac{b}{2}$	$a-\frac{3b-c}{4}$	$a-\frac{7b-4c+d}{8}$
⊐	$a-b$	$a-b+\frac{c}{2}$	$a-b+\frac{3c-d}{4}$	$a-b+\frac{7c-4d+e}{8}$
⊏	$a-b+c$	$a-b+c-\frac{d}{2}$	$a-b+c-\frac{3d-e}{4}$	$a-b+c-$ &c.
⊐	$a-b+c-d$ &c.	$a-b+$ &c.	$a-b+$ &c.	$a-b+$ &c.

Where every column consists of a set of quantities, approaching still nearer and nearer to the value of s, the terms of each column being alternately below and above that value, and each succeeding column approaching nearerthan the preceding one. Also every line, formed of all the first terms, all the second terms, all the third terms, &c, of the columns, forms also a progression whose terms continually approximate to the value of s, and each line nearer or quicker than the former; but differing from the columns, or vertical progressions, in this, namely, that whereas the terms in the columns are alternately below and above the value of s, those in each line are all on one side of the value s, namely, either all below or all above it; and the lines alternately too little and too great, namely, all the expressions in the first line too little, all those in the second line too great, those in the third line too little, and so on, every odd line being too little, and every even line too great.

7. Hence the expressions $\frac{a}{2}, \frac{3a-b}{4}, \frac{7a-4b+c}{8}$, $\frac{15a-11b+5c-a}{16}, \frac{31a-26b+16c-6d+e}{32}$, &c, are continual approximations to the value s, of the converging series $a-b+c-d+e-$ &c, and are all below the truth. But we can easily express all these several theorems by one general formula. For, since it is evident by the construction, that while the denominator in any one of them is some power (2^n) of 2 or $1+1$, the numeral co-efficients of a, b, c, &c, the terms in the numerator, are found by subtracting all the terms except the last term, one after another, from the said power 2^n or $(1+1)^n$, which is $=$

$1+n+n.\frac{n-1}{2}+n.\frac{n-1}{2}.\frac{n-2}{3}+$ &c, namely the coefficient of a equal to all the terms 2^n, minus the first term 1; that of b equal to all except the first two terms $1+n$; that of c equal to all except the first three; and so on, till the coefficient of the last term be $=1$, the last term of the power; it follows that the general expression for the several theorems, or the general approximate value of the converging series $b-a+c-d+$ &c, will be

$$\frac{2^n-1}{2^n}a-\frac{2^n-1-n}{2^n}b+\frac{2^n-1-n-n.\frac{n-1}{2}}{2^n}c+$$

&c, continued till the terms vanish and the series break off, n being equal to 0 or any integer number. Or this general formula may be expressed by this series,

$$\frac{1}{2^n}\times[(2^n-1)a-(\text{A}-n)b+(\text{B}-n.\frac{n-1}{2})c-(\text{C}-n.\frac{n-1}{2}.\frac{n-2}{3})d$$

&c]; where A, B, C, &c, denote the coefficients of the several preceding terms. And this expression, which is always too little, is the nearer to the true value of the series $a-b+c-d+$ &c, as the number n is taken greater: always excepting however those cases in which the theorem is accurately true, when n is some certain finite number. Also, with any value of n, the formula is nearer to the truth, as the terms a, b, c, &c, of the proposed series, are nearer to equality; so that the slower any proposed series converges, the more accurate is the formula, and the sooner does it afford a near value of that series: which is a very favourable circumstance, as it is in cases of very slow convergency that approximating formula are chiefly wanted. And, like as the formula approaches nearer to the truth as the terms of the series approach to an equality, so when the terms become quite equal, as in a neutral series, the formula becomes quite accurate, and always gives the same value $\frac{1}{2}a$ for s or the series, whatever integer number be taken for n. And further, when the proposed series, from being converging, passes through neutrality, when its terms are equal, and becomes diverging, the formula will still hold good, only it will then be alternately too great, and too little as long as the series diverges, as we shall presently see more fully. So that, in general, the value s of the series $a - b + c - d +$ &c, whether it be converging, diverging, or neutral, is less than the first term a; when the series converges, the value is above $\frac{1}{2}a$; when it diverges, it is below $\frac{1}{2}a$; and when neutral, it is equal to $\frac{1}{2}a$.

8. Take now the series of the first terms of the several orders of arithmetical means, which form the progression of continual approximating formulæ, being each nearer to the value of the series $a - b + c - d +$ &c, than the former, and place them in a column one under another; then take the differences between every two adjacent formulæ, and place them in another column by the side of the former, as here follows:

Approx. Formulæ.	Differences.
$\frac{a}{2}$	$\frac{a-b}{4}$
$\frac{3a-b}{4}$	$\frac{a-2b+c}{8}$
$\frac{7a-4b+c}{8}$	$\frac{a-3b+3c-d}{16}$
$\frac{15a-11b+5c-d}{16}$	$\frac{a-4b+6c-4d+e}{32}$
$\frac{31a-26b+16c-6d+e}{32}$	
&c.	&c.

From which it appears, that this series of differences consists of the very same quantities, which form the first terms of all the orders of differences of the terms of the proposed series $a-b+c-d+$ &c, when taken as usual in the differential method. And because the first of the above differences added to the first formula, gives the second formula; and the second difference added to the second formula, gives the third formula; and so on; therefore the first formula with all the differences added, will give the last formula; consequently our general formula, before mentioned,

$$\frac{1}{2^n} \times [(2^n-1)\,a-(\text{A}-n)\,b+(\text{B}-n\,.\,\frac{n-1}{2})\,c-\&\text{c}],$$

which approaches to the value of the series $a-b+c-d+$ &c, is also equivalent to, or reduces to this form,

$$\frac{a}{2}+\frac{a-b}{4}+\frac{a-2b+c}{8}+\frac{a-3b+3c-d}{16}+\ \&\text{c},$$

which, it is evident, agrees with the famous differential series. And this coincidence might be sufficient to establish the truth of our method, though we had not given other more direct proof of it. Hence it appears then, that our theorem is of the same degree of accuracy, or convergency, as the differential theorem; but admits of more direct and easy application, as the terms themselves are used, without the previous trouble of taking the several orders of differences. And our method will be rendered general for literal, as well as for numeral series, by supposing a, b, c, &c, to represent not barely the coefficients of the terms, but the whole terms, both the numeral and the literal part of them. However, as the chief use of this method is to obtain the numeral value of series, when a literal series is to be so summed, it is to be made numeral by substituting the numeral values of the letters instead of them. It is further evident, that we might easily derive our method of arithmetical means from the above differential series, by beginning with it, and receding back to our theorems, by a process counter to that above given.

9. Having, in Art. 5, 6, 7, 8, completed the investigations for the first or converging form of series, the first four articles being introductory to both forms in common; we may now proceed to the diverging form of series, for which we shall find the same method of arithmetical means, and the same general formula, as for the converging series; as well as the mode of investigation used in Art. 5 *et seq.* only changing sometimes greater for less, or less for greater. Thus then, reasoning from the table of successive sums in Art. 3, in which s is alternately above and below the expressions 0, a, $a - b$, $a - b + c$, &c, because 0 is below, and a above the value s of the series $a - b + c - d +$ &c, but 0 nearer than a to that value, it follows that s lies between 0 and $\frac{1}{2}a$, and that $\frac{1}{2}a$ is greater than s, but nearer to s than a is. In like manner, because a is above, and $a - b$ below the value s, but a nearer that value than $a - b$ is, it follows, that s lies between a and $a - b$, and that the arithmetical mean $a - \frac{1}{2}b$ is below s, but that it is nearer to s than $a - b$ is. And thus, the same reasoning holding in every pair of successive sums, the arithmetical means between them will form another series of terms, which are alternately greater and less than s, the value of the proposed series; but here greater and less in the contrary way to what they were for the converging series, namely, those steps greater here which were less there, and less here which before were greater. And this first set of arithmetical means, will either converge to the value of s, or will at least diverge less from it than the progression of successive sums. Again, the same reasoning still holding good, by taking the arithmetical means of those first means, another set is found,

which will either converge, or else diverge less than the former. And so on as far as we please, every new operation gradually checking the first or greatest divergency, till a number of the first terms of a set converge sufficiently fast, to afford a near value of s the proposed series.

10. Or, by taking the first terms of all the orders of means, we find the same set of theorems, namely

$$\frac{a}{2}, \frac{3a-b}{4}, \frac{7a-4b+c}{8}, \frac{15a-11b+5c-d}{16}, \text{ \&c, or in general,}$$

$$\frac{1}{2^n} \times [(2^n-1)a - (\text{A}-n)b + (\text{B}-n.\frac{n-1}{2})c - \text{\&c}],$$

which will be alternately above and below s, the value of the series, till the divergency is overcome. Then this series, which consists of the first terms of the several orders of means, may be treated as the successive sums, taking several orders of means of these again. After which, the first terms of these last orders may be treated again in the same manner; and so on as far as we please. Or the series of second terms, or third terms, &c, or sometimes, the terms ascending obliquely, may be treated in the same manner to advantage. And with a little practice and inspection of the several series, whether vertical, or horizontal, or oblique, for they all tend to the detection of the same value s, we shall soon learn to distinguish whereabouts the required quantity s is, and which of the series will soonest approximate to it.

11. To exemplify now this method, we shall take a few series of both sorts, and find their value, sometimes by actually going through the operations of taking the several orders of arithmetical means, and at other times by using some one of the theorems

$$\frac{a}{2}, \frac{3a-b}{4}, \frac{7a-4b+c}{8}, \frac{15a-11b+5c-d}{16}, \text{ \&c, at once.}$$

And to render the use of these theorems still easier, we shall here subjoin the following table, where the first line, consisting of the powers of 2, contains the denominators of the theorems in their order, and the figures in their perpendicular columns below them, are the coefficients of the several terms in the numerators of the theorems, namely, the upper figure, next below the power of 2, the coefficient of a; the next below, that of b; the third that of c, &c.

2	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}	2^{15}	2^{16}	2^{17}	2^{18}	2^{19}	2^{20}
1	3	7	15	31	63	127	255	511	1023	2047	4095	8191	16383	32767	65535	131071	262143	524287	1048575
	1	4	11	26	57	120	247	502	1013	2036	4083	8178	16369	32752	65519	131054	262125	524268	1048555
		1	5	16	42	99	219	466	968	1981	4017	8100	16278	32647	65399	130918	261972	524097	1048365
			1	6	22	64	163	382	848	1816	3797	7814	15914	32192	64839	130238	261156	523128	1047225
				1	7	29	93	256	638	1486	3302	7099	14913	30827	63019	127858	258096	519252	1042380
					1	8	37	130	386	1024	2510	5812	12911	27824	58651	121670	249528	507624	1026876
						1	9	46	176	562	1586	4096	9908	22819	50643	109294	230964	480492	988116
							1	10	56	232	794	2380	6476	16384	39203	89846	199140	430104	910596
								1	11	67	299	1093	3473	9949	26333	65536	155382	354522	784626
									1	12	79	378	1471	4944	14893	41226	106762	262144	616666
										1	13	92	470	1941	6885	21778	63004	169766	431910
											1	14	106	576	2517	9402	31180	94184	263950
												1	15	121	697	3214	12616	43796	137980
													1	16	137	834	4048	16664	60460
														1	17	154	988	5036	21700
															1	18	172	1160	6196
																1	19	191	1351
																	1	20	211
																		1	21
																			1

The construction and continuation of this table, is a business of little labour. For the numbers in the first horizontal line next below the line of the powers of 2, are those powers diminished each by unity. The numbers in the next horizontal line, are made from the numbers in the first, by subtracting from each the index of that power of 2 which stands above it. And for the rest of the table, the formation of it is obvious from this principle, which reigns through the whole, that every number in it is the sum of two others, namely, of the next to it on the left in the same horizontal line, and the next above that in the same vertical column. So that the whole table is formed from a few of its initial numbers, by easy operations of addition.

In converging series, it will be further useful, first to collect a few of the initial terms into one sum, and then apply our method to the following terms, which will be sooner valued, because they will converge slower.

12. For the first example, let us take the very slowly converging series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} +$ &c, which is known to express the hyp. log. of 2, which is $= \cdot 69314718$.

Here $a = 1$, $b = \frac{1}{2}$, $c = \frac{1}{3}$, $d = \frac{1}{4}$, &c, and the value, as found by theorem the 1st, 2d, 3d, 4th, 10th, and 20th, will be thus:

1st, $\dfrac{a}{2} = \frac{1}{2} = \cdot 5.$

2d, $\dfrac{3a - b}{4} = \dfrac{3 - \frac{1}{2}}{4} = \dfrac{2\frac{1}{2}}{4} = \cdot 625.$

3d, $\dfrac{7a - 4b + c}{8} = \dfrac{7 - 2 + \frac{1}{3}}{8} = \dfrac{5\frac{1}{3}}{8} = \frac{2}{3} = \cdot 666666.$

4th, $\dfrac{15a - 11b + 5c - d}{16} = \dfrac{15 - 5\frac{1}{2} + 1\frac{2}{3} - \frac{1}{4}}{16} = \cdot 68229.$

10th, $\dfrac{1023a - 1013b + \&c}{2^{10}} = \dfrac{709 \cdot 698413}{4^{5}} = \cdot 693065.$

20th, $\dfrac{1048575a - 1048555b + \&c}{2^{20}} = \dfrac{726817 \cdot 45238043}{4^{10}} =$

$\cdot 69314714.$

Where it is evident that every theorem gives always a nearer value than the former: the 10th theorem gives the value true to the 3d figure, and the 20th theorem to the 7th figure. The operation for the 10th and 20th theorems, will be easily performed by dividing, mentally, the numbers in their respective columns in the table of coefficients in Art. 11, by the ordinate numbers 1, 2, 3, 4, 5, 6, &c, placing the quotients of the alternate terms below each other, then adding each up, and dividing the difference of the sums continually five or ten times successively by the number 4: after the manner as here placed below, where the operation is set down for both of them.

1. For the 10th Theorem.

	+	−
	1023	506·5
	322·666667	212
	127·6	64·333
	25·142857	7
	1·222222	0·1
	1499·631746	789·933
	789·933333	
4	709·698413	
4	177·424603	
4	44·356151	
4	11·089038	
4	2·772259	
	·693065	

2. For the 20th Theorem.

	+	−
	1048575	524277·5
	349455	261806·25
	208476	171146·
	141159·42857143	113824·5
	87180·66666667	61666·6
	39264·54545454	21995·83333333
	10613·84615385	4318·57142857
	1446·66666667	387·25
	79·47058824	11·72222222
	1·10526316	0·05
	1886251·72936456	1159434·27698413
	1159434·27698413	
4	726817·45238043	
4	181704·36309511	
4	45426·09077378	
4	11356·52269345	
4	2839·13067336	
4	709·78266834	
4	177·44566708	
4	44·36141677	
4	11·09035419	
4	2·77258855	
	69314714	

Again, to perform the operation by taking the successive sums, and the arithmetical means: let the terms $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c, be reduced to decimal numbers, by dividing the common numerator 1 by the denominators 2, 3, 4, &c, or rather by taking these out of the table printed at the end of this volume, which contains a table of the square roots and reciprocals of all the numbers, 1, 2, 3, 4, 5, 6, &c, to 1000, and which is of great use in such calculations as these. Then the operation will stand thus:

The terms.	Suc. sums	The several orders of means.					
+ 1	1						
− 0·5	0·5						
+ 333333	833333						
− 25	583333						
+ 2	783333						
− 166666	616666	688095	692560				
+ 142857	759524	697024	693552	693056	693131		
− 125	634524	690080	692858	693205	693158	693144	693147
+ 111111	745635	695635	693362	693110	693142	693150	
− 1	645635	691090	692984	693173			
+ 090909	736544	694878					
− 083333	653211						

Here, after collecting the first twelve terms, I begin at the bottom, and, ascending upwards, take a very few arithmetical means between the successive sums, placing them on the right of them: it being unnecessary to take the means of the whole, as any part of them will do the business, but the lower ones in a converging series best, because they are nearer the value sought, and approach sooner to it. I then take the means of the first means, and the means of these again, and so on, till the value is obtained as near as may be necessary. In this process we soon distinguish whereabouts the value lies, the limits or means, which are alternately above and below it, gradually contracting, and approaching towards each other. And when the means are reduced to a single one, and it is found necessary to get the value more exactly, I then go back to the columns of successive sums, and find another first mean, either next below or above those before found, and continue it through the 2d, 3d, &c, means, which makes now a duplicate in the last column of means, and the mean between them gives another single mean of the next order; and so on as far as we see it necessary. By such a gradual progress we use no more terms nor labour than is quite requisite for the degree of accuracy required.

Or, after having collected the sum of any number of terms, we may apply any of the formulæ to the following terms. So, having as above found ·653211 for the sum of the first 12 terms, and calling the next or 13th term $\cdot 076923 = a$, the

14th term ·0714285$=b$, the next, ·06666 &c $=c$, and so on: then the 2d theorem $\frac{3a-b}{4}$ gives ·039835, which added to ·653211 the sum of the first 12 terms, gives ·693046, the value of the series true in three places of figures; and the 3d theorem $\frac{7a-4b+c}{8}$ gives ·039927 for the following terms, and which added to ·653211 the sum of the first 12 terms, gives ·693138, the value of the series true in five places. And so on.

13. For a second example, let us take the slowly converging series $\frac{2}{1}-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\frac{6}{5}-\frac{7}{6}+$ &c, which is $=\frac{1}{2}+$ hyp. log. of $2=1·19314718$. Then the process will be thus.

Terms.	Suc.sums	Arithmetical means.					
+ 2	2						
− 1·5	0·5						
+ 1·333333	1·833333						
− 1·25	0·583333						
+ 1·2	1·783333						
− 1·166666	0·616666	1·188095					
+ 1·142857	1·759524	1·197024	1·192560				
− 1·125	0·634524	1·190080	1·193552	1·193056	131		
+ 1·111111	1·745635	1·195635	1·192858	1·193205	157	144	147
− 1·1	0·645635	1·191090	1·193362	1·193110	142	150	
+ 1·090909	1·736544	1·194878	1·192984	1·193173			
− 1·083333	0·653211						

Here, after the 3d column of means, the first four figures 1·193, which are common, are omitted, to save room and the trouble of writing them so often down; and in the last three columns, the process is repeated with the last three figures of each number; and the last of these 147, joined to the first four, give 1·193147 for the value of the series proposed. And the same value is also obtained by the theorems used as in the former example.

14. For the third example, let us take the converging series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+$ &c, which is $=·7853981$ &c, or $\frac{1}{4}$ of the circumference of the circle whose diameter is 1. Here $a=1$, $b=\frac{1}{3}$, $c=\frac{1}{5}$, &c, then turning the terms into decimals, and proceeding with the successive sums and means as before, we obtain the 5th mean true within a unit in the 6th place as here below:

	Terms.	S. sums	Arithmetical means.				
+	1						
−	0·333333	666667					
+	2	866667					
−	142857	723810					
+	111111	834921					
−	090909	744012	782474				
+	76923	820935	787601	785037			
−	66667	754268	783680	785641	785339		
+	58823	813091	786775	785227	785434	785387	785397
−	52632	760459	784269	785522	785380	785407	
+	47619	808078					

15. To find the value of the converging series

$$1-\frac{1^2}{2^2}+\frac{1^2\,.\,3^2}{2^2\,.\,4^2}+\frac{1^2\,.\,3^2\,.\,5^2}{2^2\,.\,4^2\,.\,6^2}+\frac{1^2\,.\,3^2\,.\,5^2\,.\,7^2}{2^2\,.\,4^2\,.\,6^2\,.\,8^2}-\&c,$$

which occurs in the expression for determining the time of a body's descent down the arc of a circle.

The first terms of this series I find ready computed by Mr. Baron Maseres, pa. 219 Philos. Trans. 1777; these being arranged under one another, and the sums collected, &c, as before, give ·834625 for the value of that series, being only 1 too little in the last figure.

	Terms.	S. sums	Arithmetical means.				
+	1						
−	0·25	75					
+	140625	890625					
−	97656	792969					
+	74768	867737					
−	60562	807175	832620				
+	50889	858064	836124	834372			
−	43879	814185	833468	834796	834584	834618	834625
+	38565	852750	835550	834509	834652	834631	
−	34399	818351	833873	834711	834610		
+	31045	849396					

16. To find the value of $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\&c$, consisting of the reciprocals of the natural series of square numbers.

	Terms.	S. sums	Arithmetical means.					
+	1							
−	0·25	75						
+	111111	861111						
−	625	798611						
+	4	838611						
−	27778	810833						
+	20408	831241	823429					
−	15625	815616	821789	822609				
+	12346	827962	822962	822376	822492	822472		
−	1	817962	822094	822528	822452	822464	822468	822467
+	08264	826226	822754	822424	822476	822468	822466	
−	6944	819282	822240	822497	822460			
+	5917	825199						

The last mean ·822467 is true in the last figure, the more accurate value of the series $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \&c$, being ·8224670 &c.

17. Let the diverging series $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \&c$, be proposed; where the terms are the reciprocals of those in Art. 13.

Terms.	Suc. sums.	Arithmetical means.					
+ 5	+ 5						
− 666666	− 166666						
+ 75	+ 583333						
− 8	− 216666						
+ 833333	+ 616666	188095					
− 857143	− 240476	197024	192560				
+ 875	+ 634524	190080	193552	193056	131	144	
− 888889	− 254365	195635	192858	193205	157	150	147
+ 9	+ 645635	191090	193362	193110	142		
− 909091	− 263456	194878	192984	193173			
+ 916667	+ 653211						

Here the successive sums are alternately + and —, as well as the terms themselves of the proposed series, but all the arithmetical means are positive. The numbers in each column of means are alternately too great and too little, but so as visibly to approach towards each other. The same mutual approximation is visible in all the oblique lines from left to right, so that there is a general and mutual tendency, in all the columns, and in all the lines, to the limit of the value of the series. But with this difference, that all the numbers in any line descending obliquely from left to right, are on one side of the limit, and those in the next line in the same direction, all on the other side, the one line having its numbers all too great, while those in the next line are all too little; but, on the contrary, the lines which ascend from below obliquely towards the right, have their numbers alternately too great and too little, after the manner of those in the columns, but approximating quicker than those in the columns. So that, after having continued the columns of arithmetical means to any convenient extent, we may then select the terms in the last, or any other line obliquely ascending from left to right, or rather beginning with the last found mean on the right, and descending towards the left; then arrange those terms below one another in a column, and take their continual arithmetical means, like as was done with the first succesive sums, to such extent as the case may require. And if neither these new columns, nor the oblique lines approach near enough to each other, a new set may be formed from one of their oblique lines which has its terms alternately too great and too little. And thus we may proceed as far as we please. These repetitions will be more necessary in treating series which diverge more; and having here once for all described the properties attending the series, with the method of repetition, we shall only have to refer to them as occasion shall offer. In the present instance, the last two or three means vary or differ so little, that the limit may be concluded to lie nearly in the middle between them, and therefore the mean between the two last 144 and 150, namely 147, may be concluded to be very near the truth, in the last three figures; for as to the first three figures 193, repetition of them is omitted after the first three columns of means, both to save space, and the trouble of writing them so often over again. So that the value of the series in question may be concluded to be ·193147 very nearly, which is $= -\frac{1}{2} +$ the hyp. log. of 2; or 1 less than its reciprocal series in Art. 13.

18. Take the diverging series $\frac{5}{4} - \frac{5.7}{4.6} + \frac{5.7.9}{4.6.8} - \frac{5.7.9.11}{4.6.8.10} +$ &c. Here, first using some of the formulæ, we have by the

1st, $\frac{a}{2} = \cdot 625.$

2d, $\frac{3a-b}{4} = \cdot 57292.$

3d, $\frac{7a-4b+c}{8} = \cdot 56966.$

4th, $\frac{15a-11b+5c-d}{16} = \cdot 56917.$

5th, $\frac{31a-26b+16c-6d+e}{32} = \cdot 56907.$ &c.

Or, thus, taking the several orders of means, &c.

Terms.	Suc. sums.	Arithmetical means.						
+ 1·25	+ 1·25							
− 1·458333	− 0·208333	520833	566406	8685	8970	021	032	035
+ 1·640625	+ 1·432292	611980	570964	9255	9072	043	038	
− 1·804688	− 0·372396	529948	567546	8889	9015	033		
+ 1·955079	+ 1·582683	605144	570232	9141	9050			
− 2·094727	− 0·512044	535320	568050	8959				
+ 2·225647	+ 1·713603	600780	569868					
− 2·349294	− 0·635691	538956						

Here the successive sums are alternately + and −, but the arithmetical means are all +. After the second column of means, the first two figures 56 are omitted, being common; and in the last three columns the first three figures 569, which are common, are omitted. Towards the end, all the numbers, both oblique and vertical, approach so near together, that we may conclude that the last three figures 035 are all true; and these being joined to the first three 569, we have ·569035 for the value of the series, which is otherwise found $= \frac{2+\sqrt{2}}{6} =$ ·56903559 &c.

19. Let us take the diverging series

$\frac{2^2}{1} - \frac{3^2}{2} + \frac{4^2}{3} - \frac{5^2}{4} + \text{\&c}$, or $\frac{4}{1} - \frac{9}{2} + \frac{16}{3} - \frac{25}{4} + \text{\&c}$,
or $4 - 4\frac{1}{2} + 5\frac{1}{3} - 6\frac{1}{4} + 7\frac{1}{5} - 8\frac{1}{6} + \text{\&c}$.

Terms.	Sums.	Arithmetical means.					
+ 4	+ 4·						
− 4·5	− 0·5						
+ 5·333333	+ 4·833333						
− 6·25	− 1·416666						
+ 7·2	+ 5·783333						
− 8·166666	− 2·383333						
+ 9·142857	+ 6·759524	2·188096	1·942560	059	128	143	147
− 10·125	− 3·365476	1·697024	1·943557	207	158	150	
+ 11·111111	+ 7·745635	2·190080	1·942857	110	142		
− 12·1	− 4·354365	1·695635	1·943362	173			
+ 13·090909	+ 8·736544	2·191089	1·942985				
− 14·083333	− 5·346789	1·694877					

After the second column of means, the first four figures 1·943 are omitted, being common to all the following columns; to these annexing the last three figures 147 of the last mean, we have 1·943147 for the sum of the series, which we otherwise know is equal to $\frac{5}{4}$ + hyp. log. of 2. See Simp. Dissert. Ex. 2. p. 75 and 76.

And the same value might be obtained by means of the formulæ, using them as before.

20. Taking the diverging series $1 - 2 + 3 - 4 + 5 -$ &c, formed from the radix $(\frac{1}{1+1})^2 = \frac{1}{1+2+1} = \frac{1}{4}$, by dividing 1 by $1 + 2 + 1$; the method of means gives us the following.

Terms.	Sums.	Means.	
+ 1	+ 1		
− 2	− 1	0	
+ 3	+ 2	$\frac{1}{2}$	$\frac{1}{4}$
− 4	− 2	0	$\frac{1}{4}$
+ 5	+ 3	$\frac{1}{2}$	$\frac{1}{4}$
− 6	− 3	0	$\frac{1}{4}$

Where the second, and every succeeding column of means, gives $\frac{1}{4}$ for the value of the series proposed.

In like manner, using the theorems, the first gives $\frac{1}{2}$, but the second, third, fourth, &c, give each of them the same value $\frac{1}{4}$; thus:

$$\frac{a}{2} = \frac{1}{2}$$

$$\frac{3a-b}{4} = \frac{3-2}{4} = \frac{1}{4}$$

$$\frac{7a-4b+c}{8} = \frac{7-8+3}{8} = \frac{2}{8} = \frac{1}{4}$$

$$\frac{15a-11b+5c-d}{16} = \frac{15-22+15-4}{16} = \frac{4}{16} = \frac{1}{4}$$

21. Taking the series $1 - 4 + 9 - 16 + 25 - 36 +$ &c, whose terms consist of the squares of the natural series of numbers, we have, by the arithmetical means,

Terms.	Sums.	Arithmetical means.		
+ 1	+ 1			
− 4	− 3	− 1		
+ 9	+ 6	+ $1\frac{1}{2}$	+ $\frac{1}{4}$	
− 16	− 10	− 2	− $\frac{1}{4}$	0
+ 25	+ 15	+ $2\frac{1}{2}$	+ $\frac{1}{4}$	0
− 36	− 21	− 3	− $\frac{1}{4}$	0

Where it is only in the second column of means that the divergency is counteracted; after that the third and all the other orders of means give 0 for the value of the series $1 - 4 + 9 - 16 +$ &c.

The same thing takes place on using the formulæ, for

$$\frac{a}{2} = \tfrac{1}{2}$$

$$\frac{3a-b}{4} = \frac{3-4}{4} = -\frac{1}{4}$$

$$\frac{7a-4b+c}{8} = \frac{7-16+9}{8} = \frac{0}{8} = 0$$

$$\frac{15a-11b+5c-d}{16} = \frac{15-44+45-16}{16} = \frac{0}{16} = 0$$

where the third and all after it give the same value 0.

22. Taking the geometrical series of terms $1-2+4-8+$ &c, derived from the radix $\frac{1}{1+2} = \frac{1}{3}$, by actually dividing 1 by $1+2$.

Terms.	Sums.	Arithmetical means.				
+ 1	+ 1	+ ½	+ ¼	⅜	$\frac{5}{16}$	
− 2	− 1	0	+ ½	¼	⅜	
+ 4	+ 3	+ 1	0	½	¼	
− 8	− 5	− 1	+ 1	0	½	&c.
+ 16	+ 11	+ 3	− 1	+ 1	0	
− 32	− 21	− 5	+ 3	− 1	+ 1	
+ 64	+ 43	+ 11	− 5	+ 3	− 1	
− 128	− 85	− 21	+ 11	− 5	&c.	
+ 256	+ 171	+ 43				

Here the lower parts of all the columns of means, from the cipher 0 downwards, consist of the same series of terms $+1-1+3-5+11-21+43-85+$ &c, and the other part of the columns, from the cipher upwards, as well as each line of oblique means, parallel to, and above the line of ciphers, forms a series of terms $\frac{1}{2}, \frac{1}{4}, \frac{3}{8}, \frac{5}{16} \; . \; . \; . \; . \; \frac{1}{3} \cdot \frac{2^n \pm 1}{2^n}$, alternately above and below the value of the series, $\frac{1}{3}$, and approaching continually nearer and nearer to it, and which, when infinitely continued, or when n is infinite, the term becomes $\frac{1}{3}$ for the value of the geometrical series, $1-2+4-8+16-$ &c.

And the same set of terms would be given by each of the formulæ.

23. Taking the geometrical series $1-3+9-27+81-$ &c, obtained from the radix $\frac{1}{1+3}=\frac{1}{4}$, by dividing 1 by $1+3$.

Terms.	Sums.	Arithmetical means.							
$+1$	$+1$	$+\frac{1}{2}$	0	$+\frac{1}{2}$	0	$+\frac{1}{2}$	0	$+\frac{1}{2}$	0
-3	-2	$-\frac{1}{2}$	$+1$	$-\frac{1}{2}$	$+1$	$-\frac{1}{2}$	$+1$	$-\frac{1}{2}$	
$+9$	$+7$	$+2\frac{1}{2}$	-2	$+2\frac{1}{2}$	-2	$+2\frac{1}{2}$	-2		
-27	-20	$-6\frac{1}{2}$	$+7$	$-6\frac{1}{2}$	$+7$				
$+81$	$+61$	$+20\frac{1}{2}$	-20						

Here the column of successive sums, and every second column of the arithmetical means, below the 0, consists of the same series of terms $1, -2, +7, -20, +$ &c, while all the other columns of means consist of this other set of terms $\frac{1}{2}, -\frac{1}{2}, +2\frac{1}{2}, -6\frac{1}{2}, +$ &c; also the first oblique line of means, $\frac{1}{2}$, 0, $\frac{1}{2}$, 0, $\frac{1}{2}$, 0, &c, consists of the terms $\frac{1}{2}$ and 0 alternately, which are all at equal distance from the value of the series proposed $1-3+9-27+81-$ &c, as indeed are the terms of all the other oblique descending lines. And the mean between every two terms gives $\frac{1}{4}$ for that value. And the same terms would be given by the formulæ, namely alternately $\frac{1}{2}$ and 0.

And thus the value of any geometrical series, whose ratio or second term is r, will be found to be $=\frac{1}{1+r}$.

24. Finally, let there be taken the hypergeometrical series $1-1+2-6+24-120+$ &c $=1-1\text{A}+2\text{B}-3\text{C}+4\text{D}-5\text{E}+$ &c; which difficult series has been honoured by a very considerable memoir written on the valuation of it by the celebrated L. Euler, in the New Petersburg Commentaries, vol. v, where the value of it is at length determined to be ·5963473 &c.

To simplify this series, let us omit the first two terms $1-1=0$, which will not alter the value, and divide the remaining terms by 2, and the quotients will give $1-3+12-60+360-2520+$ &c; which, being half the proposed series, ought to have for its value the half of ·596347 &c, namely ·298174 nearly.

Now, ranging the terms in a column, and taking the sums and means as usual, we have

Terms.	Sums.	Arithmetical means.						
+ 1	+ 1	+ $\frac{1}{2}$						
− 3	− 2	− $\frac{1}{2}$	0	$\frac{7}{8}$				
+ 12	+ 10	+ 4	$1\frac{3}{4}$	$3\frac{1}{8}$	1·1250	4·53125		
− 60	− 50	− 20	8	$23\frac{1}{2}$	10·1875	33·40625	14·4375	59·023438
+ 360	+ 310	+ 130	55	$177\frac{1}{2}$	77·	298·375	132·484375	
− 2520	− 2210	− 950	410	1525	673·75			
+ 20160	+ 17950	+ 7870	3460					

Where it is evident, that the diverging is somewhat diminished, but not quite counteracted, in the columns and oblique descending lines, from beginning to end, as the terms in those directions still increase, though not quite so fast as the original series; and that the signs of the same terms are alternately + and −, while those of the terms in the other lines obliquely ascending from left to right, are alternately one line all +, and another line all −, and these terms continually decreasing. The terms in the oblique descending lines, being alternately too great and too little, are the fittest to proceed with again. Taking therefore any one of those lines, as suppose the first, and ranging it vertically, take the means as before, and they will approach nearer to the value of the series, thus:

+ ·5						
− ·0	+ ·25	+ ·34375				
+ ·875	+ ·4375	+ ·15625	+ ·25	+ ·361328		
− 1·125	− ·1250	+ ·789062	+ ·472656	+ ·027344	·194336	·492066
+ 4·53125	+ 1·703125	− 1·625	− ·417969	+ 1·552246	·789795	
− 14·4375	− 4·953125	+ 8·669922	+ 3·522461			
+ 59·023438	+ 22·292969					

Here the same approximation in the lines and columns, towards the value of the series, is observable again, only in a higher degree; also the terms in the columns and oblique descending lines, are again alternately too great and too little, but now within narrower limits, and the signs of the terms are more of them positive; also the terms in each oblique ascending line, are still either all above or all below the value of the series, and that alternately one line after another, as before. But the descending lines will again be the fittest to use, because the terms in each are alternately above and below the value sought. Taking therefore again the first of these oblique descending lines, and treating it as before, we obtain sets of terms approaching still nearer to the value, thus:

25					
34375	296875				
25	296875	296875			
361328	305664	301271	299073		
194336	277832	291748	296509	297791	
492066	343201	310516	301132	298821	298306

Here the approach to an equality, among all the lines and columns, is still more visible, and the deviations restricted within narrower limits, the terms in the oblique ascending lines still on one side of the value, and gradually increasing, while the columns and the oblique descending lines, for the most part, have their terms alternately too great and too little, as is evident from their alternately becoming greater and less than each other: and from an inspection of the whole, it is easy to pronounce that the first three figures of the number sought, will be 298. Taking therefore the last four terms of the first descending line, and proceeding as before, we have

296875			
299073	297974		
297791	298432	298203	
298306	298048	298240	298222

And, finally, taking the lowest ascending line, because it has most the appearance of being alternately too great and too little, proceed with it as before, thus:

298306			
298048	298177		
298240	298144	298161	
298222	298231	298187	298174

where the numbers in the lines and columns gradually approach nearer together, till the last mean is true to the nearest unit in the last figure, giving us ·298174 for the value of the proposed hypergeometrical series $1 - 3 + 12 - 60 + 360 - 2520 + 20160 -$ &c.

And in like manner are we to proceed with any other series whose terms have alternate signs.

Royal Military Academy,
Woolwich, May, 1780.

POSTSCRIPT.

Since the foregoing method was discovered, and made known to several friends, two passages have been offered to my consideration, which I shall here mention, in justice to their authors, Sir I. Newton, and the late learned Mr. Euler.

The first of these is in Sir Isaac's letter to Mr. Oldenburg, dated October 24, 1676, and may be seen in Collins's *Commercium Epistolicum*, p. 177, the last paragraph near the bottom of the page, namely, *Per seriem Leibnitii etiam*, si ultimo loco dimidium termini adjiciatur, *et alia quædam similia artificia adhibeantur, potest computum produci ad multas figuras.* The series here alluded to, is $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} +$ &c, denoting the area of the circle whose diameter is 1; and Sir Isaac here directs to add in half the last term, after having collected all the foregoing, as the means of obtaining the sum a little exacter. And this, indeed, is equivalent to taking one arithmetical mean between two successive sums, but it does not reach the idea contained in my method. It appears also, both by the other words, *et alia quædam similia artificia adhibeantur*, contained in the above extract, and by these, *alias artes adhibuissem*, a little higher up in the same pa. 177, that Sir Isaac Newton had several other contrivances for obtaining the sums of slowly converging series; but what they were, it may perhaps be now impossible to determine.

The other is a passage in the *Novi Comment. Petropol.* tom. v. p. 226, where Mr. Euler gives an instance of taking one set of arithmetical means between a series of quantities which are gradually too little and too great, to obtain a nearer value of the sum of a series in question. But neither does this reach the idea contained in our method. However, I have thought it but justice to the characters of these two eminent men, to make this mention of their ideas, which have some relation to my own, though unknown to me at the time of my discovery.