# TRACT VII.

## A DISSERTATION ON THE NATURE AND VALUE OF INFINITE SERIES.

1. About the year 1780 I discovered a very general and easy method of valuing series, whose terms are alternately positive and negative, which equally applies to such series, whether they be converging, or diverging, or their terms all equal; together with several other properties relating to certain series: and as there may be occasion to deliver some of those matters in the course of these tracts, this opportunity is taken of premising a few ideas and remarks, on the nature and valuation of some of the classes of series, which form the object of those communications. This is done with a view to obviate any misconceptions that might perhaps be made, concerning the idea annexed to the term *value* of such series in those tracts, and the sense in which it is there always to be understood; which is the more necessary, as many controversies have been warmly agitated concerning these matters, not only of late, by some of our own countrymen, but also by others among the ablest mathematicians in Europe, at different periods in the course of the last century; and all this, it seems, through the want of specifying in what sense the term *value* or *sum* was to be understood in their dissertations. And in this discourse, I shall follow, in a great measure, the sentiments and manner of the late celebrated L. Euler, contained in a similar memoir of his in the fifth volume of the New Petersburgh Commentaries, adding and intermixing here and there other remarks and observations of my own.

2. By a converging series, is meant such a one whose terms continually decrease; and by a diverging series, that

whose terms continually increase. So that a series whose terms neither increase nor decrease, but are all equal, as they neither converge nor diverge, may be called a neutral series, as $a - a + a - a +$ &c. Now converging series, being supposed infinitely continued, may have their terms decreasing to 0 as a limit, as the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} +$ &c, or only decreasing to some finite magnitude as a limit, as the series $\frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} +$ &c, which tends continually to 1 as a limit. So, in like manner, diverging series may have their terms tending to a limit, that is either finite or infinitely great: thus the terms $1 - 2 + 3 - 4 +$ &c, diverge to infinity; but the diverging terms $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} +$ &c, only to the finite magnitude 1. Hence then, as the ultimate terms of series which do not converge to 0, by supposing them continued *in infinitum*, may be either finite or infinite, there will be two kinds of such series, each of which will be further divided into two species, according as the terms shall either be all affected with the same sign, or have alternately the signs + and −. We shall, therefore, have altogether four species of series which do not converge to 0, an example of each of which may be as here follows:

$$1. \quad \begin{cases} 1 + 1 + 1 + 1 + 1 + 1 + \text{\&c.} \\ \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \frac{6}{7} + \text{\&c.} \end{cases}$$

$$2. \quad \begin{cases} 1 - 1 + 1 - 1 + 1 - 1 + \text{\&c.} \\ \frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \frac{5}{6} - \frac{6}{7} + \text{\&c.} \end{cases}$$

$$3. \quad \begin{cases} 1 + 2 + 3 + 4 + 5 + 6 + \text{\&c.} \\ 1 + 2 + 4 + 8 + 16 + 32 + \text{\&c.} \end{cases}$$

$$4. \quad \begin{cases} 1 - 2 + 3 - 4 + 5 - 6 + \text{\&c.} \\ 1 - 2 + 4 - 8 + 16 - 32 + \text{\&c.} \end{cases}$$

3. Now concerning the sums of these species of series, there have been great dissensions among mathematicians; some affirming that they can be expressed by a certain sum, while others deny it. In the first place, however, it is evident that the sums of such series as come under the first of these species, will be really infinitely great, since by actually

collecting the terms, we can arrive at a sum greater than any proposed number whatever: and hence there can be no doubt but that the sums of this species of series may be exhibited by expressions of this kind $\frac{a}{0}$. It is concerning the other species, therefore, that mathematicians have chiefly differed; and the arguments which both sides allege in defence of their opinions, have been endued with such force, that neither party could be hitherto brought to yield to the other.

4. As to the second species, the celebrated Leibnitz was one of the first who treated of this series $1 - 1 + 1 - 1 + 1 - 1 +$ &c, and he concluded the sum of it to be $= \frac{1}{2}$, relying on the following cogent reasons. And first, that this series arises by resolving the fraction $\frac{1}{1+a}$ into the series $1 - a + a^2 - a^3 + a^4 - a^5 +$ &c, by continual division in the usual way, and taking the value of $a$ equal to unity. Secondly, for more confirmation, and for persuading such as are not accustomed to calculations, he reasons in the following manner: If the series terminate any where, and if the number of the terms be even, then its value will be $= 0$; but if the number of terms be odd, the value of the series will be $= 1$: but because the series proceeds *in infinitum*, and that the number of the terms cannot be reckoned either odd or even, we may conclude that the sum is neither $= 0$, nor $= 1$, but that it must obtain a certain middle value, equidifferent from both, and which is therefore $= \frac{1}{2}$. And thus, he adds, nature adheres to the universal law of justice, giving no partial preference to either side.

5. Against these arguments the adverse party make use of such objections as the following. First, that the fraction $\frac{1}{1+a}$ is not equal to the infinite series $1 - a + a^2 - a^3 +$ &c, unless $a$ be a fraction less than unity. For if the division be any where broken off, and the quotient of the remainder be added, the cause of the paralogism will be manifest;

for we shall then have $\frac{1}{1+a} = 1 - a + a^2 - a^3 + \pm a^n \mp \frac{a^{n+1}}{1+a}$; and that, although the number $n$ should be made infinite, yet the supplemental fraction $\mp \frac{a^{n+1}}{1+a}$ ought not to be omitted, unless it should become evanescent, which happens only in those cases in which $a$ is less than 1, and the terms of the series converge to 0. But that in other cases there ought always to be included this kind of supplement $\mp \frac{a^{n+1}}{1+a}$; and though it be affected with the dubious sign $\mp$, namely $-$ or $+$ according as $n$ shall be an even or an odd number, yet if $n$ be infinite, it may not therefore be omitted, under the pretence that an infinite number is neither odd nor even, and that there is no reason why the one sign should be used rather than the other; for it is absurd to suppose that there can be any integer number, even though it be infinite, which is neither odd nor even.

6. But this objection is rejected by those who attribute determinate sums to diverging series, because it considers an infinite number as a determinate number, and therefore either odd or even, when it is really indeterminate. For that it is contrary to the very idea of a series, said to proceed *in infinitum*, to conceive any term of it as the last, though infinite: and that therefore the objection above-mentioned, of the supplement to be added or subtracted, naturally falls of itself. Therefore, since an infinite series never terminates, we never can arrive at the place where that supplement must be joined; and therefore that the supplement not only may, but indeed ought to be neglected, because there is no place found for it.

And these arguments, adduced either for or against the sums of such series as above, hold also in the fourth species, which is not otherwise embarrassed with any further doubts peculiar to itself.

7. But those who dispute against the sums of such series,

think they have the firmest hold in the third species. For though the terms of these series continually increase, and that, by actually collecting the terms, we can arrive at a sum greater than any assignable number, which is the very definition of infinity; yet the patrons of the sums are forced to admit, in this species, series whose sums are not only finite, but even negative, or less than nothing. For since the fraction $\frac{1}{1-a}$, by evolving it by division, becomes $1 + a + a^2 + a^3 + a^4 +$ &c, we should have

$$\frac{1}{1-2} = -1 = 1 + 2 + 4 + 8 + 16 + \&c,$$

$$\frac{1}{1-3} = -\tfrac{1}{2} = 1 + 3 + 9 + 27 + 81 + \&c,$$

which their adversaries, not undeservedly, hold to be absurd, since by the addition of affirmative numbers, we can never obtain a negative sum; and hence they urge that there is the greater necessity for including the before-mentioned supplement additive, since by taking it in, it is evident that

$$-1 \text{ is } = 1 + 2 + 4 + 8 \ \ldots\ldots\ 2^n + \frac{2^{n+1}}{1-2},$$

though $n$ should be an infinite number.

8. The defenders therefore of the sums of such series, in order to reconcile this striking paradox, more subtle perhaps than true, make a distinction between negative quantities; for they argue, that while some are less than nothing, there are others greater than infinite, or above infinity. Namely, that the one value of $-1$ ought to be understood, when it is conceived to arise from the subtraction of a greater number $a + 1$ from a less $a$; but the other value, when it is found equal to the series $1 + 2 + 4 + 8 +$ &c, and arising from the division of the number 1 by $-1$; for that in the former case it is less than nothing, but in the latter greater than infinite. For the more confirmation, they bring this example of fractions

$$\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{1}, \frac{1}{0}, \frac{1}{-1}, \frac{1}{-2}, \frac{1}{-3}, \&c,$$

which, evidently increasing in the leading terms, it is inferred will continually increase; and hence they conclude that $\frac{1}{-1}$ is greater than $\frac{1}{0}$, and $\frac{1}{-2}$ greater than $\frac{1}{-1}$, and so on: and therefore as $\frac{1}{-1}$ is expressed by $-1$, and $\frac{1}{0}$ by $\infty$, or infinity, $-1$ will be greater than $\infty$, and much more will $= \frac{1}{-\frac{1}{2}}$ be greater than $\infty$. And thus they ingeniously enough repelled that apparent absurdity by itself.

9. But though this distinction seemed to be ingeniously devised, it gave but little satisfaction to the adversaries; and besides, it seemed to affect the truth of the rules of algebra. For if the two values of $-1$, namely $1-2$ and $\frac{1}{-1}$, be really different from each other, as we may not confound them, the certainty and the use of the rules, which we follow in making calculations, would be quite done away; which would be a greater absurdity than that for whose sake the distinction was devised: but if $1 - 2 = \frac{1}{-1}$, as the rules of algebra require, for by multiplication $-1 \times (1-2) = -1 + 2 = 1$, the matter in debate is not settled; since the quantity $-1$, to which the series $1 + 2 + 4 + 8 +$ &c, is made equal, is less than nothing, and therefore the same difficulty still remains. In the mean time however, it seems but agreeable to truth, to say, that the same quantities which are below nothing, may be taken as above infinite. For we know, not only from algebra, but from geometry also, that there are two ways, by which quantities pass from positive to negative, the one through the cypher or nothing, and the other through infinity: and besides, that quantities, either by increasing or decreasing from the cypher, return again, and revert to the same term 0; so that quantities more than infinite are the same with quantities less than nothing, like as quantities less than infinite agree with quantities greater than nothing.

10. But, further, those who deny the truth of the sums

that have been assigned to diverging series, not only omit to assign other values for the sums, but even set themselves utterly to oppose all sums whatever belonging to such series, as things merely imaginary. For a converging series, as suppose this $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$ &c, will admit of a sum $= 2$, because the more terms of this series we actually add, the nearer we come to the number 2: but in diverging series the case is quite different; for the more terms we add, the more do the sums which are produced differ from one another, neither do they ever tend to any certain determinate value. Hence they conclude, that no idea of a sum can be applied to diverging series, and that the labour of those persons who employ themselves in investigating the sums of such series, is manifestly useless, and indeed contrary to the very principles of analysis.

11. But notwithstanding this seemingly real difference, yet neither party could ever convict the other of any error, whenever the use of series of this kind has occurred in analysis; and for this good reason, that neither party is in an error, the whole difference consisting in words only. For if in any calculation we arrive at this series $1 - 1 + 1 - 1 +$ &c, and that we substitute $\frac{1}{2}$ instead of it, we shall surely not thereby commit any error; which however we should certainly incur if we substitute any other number instead of that series; and hence there remains no doubt but that the series $1 - 1 + 1 - 1 +$ &c, and the fraction $\frac{1}{2}$, are equivalent quantities, and that the one may always be substituted instead of the other without error. So that the whole matter in dispute seems to be reduced to this only, namely, whether the fraction $\frac{1}{2}$ can be properly called the *sum* of the series $1 - 1 + 1 - 1 +$ &c. Now if any persons should obstinately deny this, since they will not however venture to deny the fraction to be equivalent to the series, it is greatly to be feared they will fall into mere quarrelling about words.

12. But perhaps the whole dispute will easily be compromised, by carefully attending to what follows. Whenever, in analysis, we arrive at a complex function or expression,

either fractional or transcendental; it is usual to convert it into a convenient series, to which the remaining calculus may be more easily applied. And hence the occasion and rise of infinite series. So far only then do infinite series take place in analytics, as they arise from the evolution of some finite expression; and therefore, instead of an infinite series, in any calculus, we may substitute that formula, from whose evolution it arose. And hence, for performing calculations with more ease or more benefit, like as rules are usually given for converting into infinite series such finite expressions as are endued with less proper forms; so, on the other hand, those rules are to be esteemed not less useful, by the help of which we may investigate the finite expression from which a proposed infinite series would result, if that finite expression should be evolved by the proper rules: and since this expression may always, without error, be substituted instead of the infinite series, they must necessarily be of the same value: and hence no infinite series can be proposed, but a finite expression may, at the same time, be conceived as equivalent to it.

13. If, therefore, we only so far change the received notion of a sum as to say, that the sum of any series, is the finite expression by the evolution of which that series may be produced, all the difficulties, which have been agitated on both sides, vanish of themselves. For, first, that expression by whose evolution a converging series is produced, exhibits at the same time its sum, in the common acceptation of the term: neither, if the series should be divergent, could the investigation be deemed at all more absurd, or less proper, namely, the searching out a finite expression which, being evolved according to the rules of algebra, shall produce that series. And since that expression may be substituted in the calculation instead of this series, there can be no doubt but that it is equal to it. Which being the case, we need not necessarily deviate from the usual mode of speaking, but might be permitted to call that expression also the *sum*, which is *equal* to any series whatever, provided however,

that, in series whose terms do not converge to 0, we do not connect that notion with this idea of a sum, namely, that the more terms of the series are actually collected, the nearer we must approach to the value of the sum.

14. But if any person shall still think it improper to apply the term sum, to the finite expressions by whose evolution all series in general are produced; it will make no difference in the nature of the thing; and instead of the word sum, for such finite expression, he may use the term value, or function, or perhaps the term *radix* would be as proper as any other that could be employed for this purpose, as the series may justly be considered as issuing or growing out of it, like as a plant springs from its root, or from its seed. The choice of terms being in a great measure arbitrary, every person is at liberty to employ them in whatever sense he may think fit, or proper for the purpose in hand; provided always that he fix and determine the sense in which he understands or employs them. And as I consider any series, and the finite expression by whose evolution that series may be produced, as no more than two different ways of expressing one and the same thing, whether that finite expression be called the sum, or value, or function, or radix of the series; so in the following paper, and in some others which may perhaps hereafter be produced, it is in this sense I desire to be understood, when searching out the value of series, namely, that the object of the enquiry, is the radix by whose evolution the series may be produced, or else an approximation to the value of it in decimal numbers, &c.