TRACT I.

THE

PRINCIPLES OF BRIDGES:

CONTAINING

THE MATHEMATICAL DEMONSTRATION OF THE PROPERTIES OF THE ARCHES, THE THICKNESS OF THE PIERS, THE FORCE OF THE WATER AGAINST THEM, &c. WITH PRACTICAL OBSERVATIONS AND DIRECTIONS DRAWN FROM THE WHOLE.

THIS Tract, on bridges, originated from the circumstance of the fall of Newcastle bridge, in the year 1771; which, with other particulars relative to the Tract, are noticed in the Preface to that Edition of it; which was as follows:

THE ORIGINAL PREFACE.

A large and elegant bridge, forming a way over a broad and rapid river, is justly esteemed one of the noblest pieces of mechanism that man is capable of performing. And the usefulness of an art which, at the same time that it connects distant shores by a way over the deep and rapid waters, also allows those waters and their navigation to pass smooth and uninterrupted, renders all probable attempts to advance the theory or practice of it, highly deserving the encouragement of the public.

This little book is offered as an attempt towards the improvement of the theory of this art, in which the more essential properties, dimensions, proportions, and other relations of the various parts of a bridge, are strictly demonstrated, and clearly illustrated by various examples. It is divided into five sections: the 1st treats on the projects of bridges, containing a regular detail of the various circumstances and considerations that are cognizable in such projects. The 2d treats on arches, demonstrating their various properties, with the relations between their intrados and extrados, and clearly distinguishing the most preferable curves to be used in a bridge; the first two or three propositions being instituted after the manner of two or three done by Mr. Emerson in his Fluxions and Mechanics. The 3d section treats on the piers, demonstrating their thickness necessary for supporting any kind of an arch, springing at any height, both when part of the pier is supposed to be immersed in water, and when otherwise. The 4th demonstrates the force of the water against the end or face of the pier, considered as of different forms; with the best form for dividing the stream, &c. and to it is added a table, showing the several heights of the fall of the water under the arches, arising from its velocity and the obstruction of the piers; as it was composed by Tho. Wright, Esq. of Auckland, in the county of Durham, who informs me it is part of a work on which he has spent much time, and with which he intends to favour the public. And the 5th and last section contains a Dictionary of the most material terms relating to the subject: in which many practical observations and directions are given, which could not be so regularly nor properly introduced into the former sections. The whole, it is presumed, containing full directions for constituting and adapting to one another, the several essential parts of a bridge, so as to make it the strongest, and the most convenient, both for the passage over and under it, which the situation and other circumstances will admit: not indeed for the actual methods of disposing the stones, making of mortar, or the external ornaments, &c. those things are not here attempted, but are left to the discretion of the practical architect, as being no part of the plan of this undertaking; and for the same reason also here are not given any views of bridges, but only prints of such parts or figures as are necessary in explaining the elementary parts of the subject.

As my profession is not that of an architect, very probably I should never have turned my thoughts to this subject, so as to address the public upon it, had it not been for the occasion of an accident in that part of the country in which I reside, viz. the fall of Newcastle and other bridges on the river Tyne, on the 17th of November, 1771, occasioned by a high flood, which rose about 9 feet higher at Newcastle than the usual spring tides do. This occasion having furnished me with many opportunities of hearing and seeing very absurd notions advanced on the subject in general, I thought the demonstrations of the relations of the essential parts of a bridge, would not be unacceptable to those architects and others, who may be capable of perceiving their force and effects.

Newcastle, 1772.

The original edition, of 1772, being out of print, and the book being much asked for, a new edition was printed in 1801, at a time when the project of a cast-iron bridge of one arch, proposed to be built over the Thames at London, by Messrs. Telford and Douglass, was the subject of much conversation: on which occasion the following addition was made to the Preface; viz.

This little work, which was hastily composed on a particular occasion, having been long out of print, is now as suddenly reprinted in the same form, on the present occasion, of the report of a new bridge proposed to be thrown across the Thames, at London: reserving the long intended edition, on a much larger and more improved plan, till a more convenient opportunity.

Royal Military Academy, Jan. 12, 1801.

It may here be added, that the whole tract has been now quite re-cast and composed, and greatly enlarged with more propositions, and numerous observations, both practical and scientific. To the end is also added an Appendix, being the author's report to the Committee of Parliament, on the project for a new cast-iron bridge, of one arch, over the river at London; and several other appropriate appendages.

SECTION I.

ON THE PROJECTS OF BRIDGES; WITH THE DESIGN, THE ESTIMATE, &c.

WHEN a bridge is deemed necessary to be built over a river, the first consideration is the place of it; or what particular situation will contain a maximum of the advantages over the disadvantages. In agitating this important question, every circumstance, certain and probable, attending or likely to attend the bridge, should be separately, minutely, and impartially stated and examined; and the advantage or disadvantage of it rated at a value proportioned to it; then the difference between the whole advantages and disadvantages, will be the net value of that particular situation for which the calculation is made. And by doing the same for other situations, all their net values will be found, and of consequence the most preferable situation among them.—Or, in a competition between two places, if each one's advantage over the other be estimated or valued in every circumstance attending them, the sums of their advantages will show which of them is the better. And the same being done for this and a third, and so on, the best situation of all will be obtained.

In this estimation, a great number of particulars must be included; nothing being omitted that can be found to make a part of the consideration. Among these, the situation of the town or place, for the convenience of which the bridge is chiefly to be made, will naturally produce an article of the first consequence; and a great many others, if necessary, ought to be sacrificed to it. If possible, the bridge should be placed where there can conveniently be opened and made passages or streets from the end of it in every direction, and especially one as nearly in the direction of the bridge itself as possible, tending towards the body of the town, without narrows or crooked windings, and easily communicating with the chief streets, thoroughfares, &c.—And here every person, in judging of this, should divest himself of all partial regards or attachments whatever; think and determine for the good of the whole only, and for posterity as well as for the present.

The banks or declivities towards the river are also of particular concern, as they affect the conveniency of the passage to and from the bridge, or determine the height of it, on which in a great measure depends the expense, as well as the convenience of passage. The breadth of the river, the navigation upon it, and the quantity of water to be passed, or the velocity and depth of the stream, form also considerations of great moment; as they determine the bridge to be higher or lower, longer or shorter. However, in most cases, a wide part of the river ought rather to be chosen than a narrow one, especially if it is subject to great tides or floods: for, the increased velocity of the stream in the narrow part, being again augmented by the further contraction of the breadth by the piers of the bridge, will both incommode the navigation through the arches, and undermine the piers and endanger the whole bridge. The nature of the bed of the river is also of great concern, it having a great influence on the expense; as upon it, and the depth and velocity of the stream, depend the manner of laying the foundations, and building the piers. These are the chief and capital articles of consideration, which will branch themselves out into other dependent ones, and so lead to the required estimate of the whole.

Having resolved on the place, the next considerations are, the form, the estimate of the expense, and the manner of execution. With respect to the form; strength, utility, and beauty ought to be regarded and united; the chief part of which lies in the arches. The form of the arches will depend on their height and span; and the height on that of the water, the navigation, and the adjacent banks. They ought to be made so high, as that they may easily transmit the water at its greatest height, either from tides or floods; and their height and figure ought also to be such as will easily allow of a convenient passage of the craft through them. This, and the disposition of the bridge above, so as to render the passage over it also convenient, make up its utility.—Having fixed the heights of the arches, their spans are still necessary for determining their figure. Their spans will be known by dividing the whole breadth of the river into a convenient number of arches and piers, allowing at least the necessary thickness of the piers out of the whole. In fixing on the number of arches, let an odd number always be taken; and few and large ones, rather than many and smaller, if convenient: For thus we shall have not only fewer foundations and piers to make, but fewer arches and centres, which will produce great savings in the expense; and besides, the arches themselves will also require much less materials and workmanship, and allow of more and better passage for the water and craft through them; and will appear at the same time more noble and graceful, especially if constructed in elliptical, or in cycloidal forms; for the truth of which, it may be sufficient to refer to that noble and elegant bridge lately built at Blackfriars, London, by Mr. Mylne; which might perhaps be accounted incomparable, at least in England, if the piers were of equal excellence: but these are too thick, and clumsy, and their appearance is made still less graceful by the double columns placed before them. So that Blackfriar's arches and the Westminster's piers united, would be preferable to either bridge separately.

If the top of the bridge be a straight horizontal line, the arches may be made all of a size; if it be a little lower at the ends than the middle, the arches must proportionally decrease from the middle towards the ends; but if higher at the ends than the middle, which can seldom happen, they may then increase towards the ends. A choice of the most convenient arches is to be made from some of the following propositions, where their several properties and effects are demonstrated and pointed out: Among these, the elliptic, cycloidal, and equilibrate arch, will generally claim the preference, as well on account of the strength, and beauty, as cheapness or saving in materials and labour: Other particulars also concerning them may be seen under the word ARCH in the Dictionary in the last section.

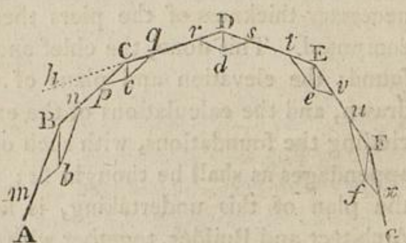
Next find what thickness at the keystone or top will be necessary for the arches. For which see the word KEYSTONE in the Dictionary in the 5th section.—Having thus obtained all the parts of the arches, with the height of the piers, the necessary thickness of the piers themselves are next to be computed. This done, the chief and material requisites are found; the elevation and plans of the design can then be drawn, and the calculations of the expense thence made, including the foundations, with such ornamental or accidental appendages as shall be thought fit; which, being no part of the plan of this undertaking, is left to the fancy of the Architect and Builder, together with the practical methods of carrying the design into execution. I shall however, in the Dictionary, in the last section, not only describe the terms, parts, machines, &c, but also speak of their dimensions, properties, and any thing else material belonging to them; and to which therefore I from hence refer for more explicit information in each particular article, as well as to these immediately following propositions, in which the theory of the arches, piers, &c, are fully and strictly demonstrated.

SECTION II.

OF THE ARCHES.

PROPOSITION I.

Let there be any number of lines AB, BC, CD, DE, *&c. all in the same vertical plane, connected together and moveable about the joints or angles* A, B, C, D, E, F; *the two extreme points* A *and* G *being fixed: It is required to determine the proportion of the weights to be laid upon the angles* B, C, D, *&c. so that the whole may remain in equilibrio.*



Solution.—From the several angles, having drawn the lines Bb, Cc, Dd, &c. perpendicular to the horizon; about them, as diagonals, constitute parallelograms such, that those sides of every two that are at the opposite ends of the given lines, may be equal to each other; viz, having made one parallelogram mn, take Cp = Bn, and form the parallelogram pq; then take Dr = Cq, and make the parallelogram rs; and take Er = Ds, and form the parallelogram tv; and so on: Then the said vertical diagonals Bb, Cc, Dd, Ee, &c, of those parallelograms, will be proportional to the weights, as required.

Demonstration.—By the resolution of forces, each of the weights or forces Bb, Cc, Dd, &c, in the diagonals of the parallelograms, is equal to, and may be resolved into, two forces, expressed by two adjacent sides of the parallelogram; viz. the force Bb may be resolved into the two forces Bm, Bn, and in those directions; the force C*c*, into the two forces C*p*, C*q*, and in those directions; the force D*d*, into the forces D*r*, D*s*, and in those directions; and so on. Then, since two forces that are equal, and in opposite directions, do mutually balance each other; therefore the several pairs of forces B*n* and C*p*, C*q* and D*r*, D*s* and E*t*, &c, being equal and opposite, by the construction, mutually destroy or balance each other; and the extreme forces B*m*, E*v*, are balanced by the opposite resistances of the fixed points A, G. There is no force therefore to change the position of any one of the lines, and consequently they will all remain in equilibrio.

Corollary.—Hence, if one of the weights and the positions of all the lines be given, all the other weights may thence be found, as well as all the oblique forces in the direction of the bars or lines. And the weight which is given, may either be that at the lower extremity, as B*b*, or it may be that at the vertex D*d*, or it may be any of the intermediate ones, as C*c*: for, whichever of these is given, it will serve, as a diagonal, to form the parallelogram about it; then the sides of this parallelogram will give the sides of the two next parallelograms, on each side of the former; and so on through the whole collection of the bars. Thus, if the uppermost vertical weight, or diagonal D*d*, be the given one: Then draw *dr* parallel to DE, and *ds* to DC, so forming the parallelogram *r*D*sd*: then make C*q* = D*r*, and E*t* = D*s*: and, having drawn the several indefinite vertical lines B*b*, C*c*, E*e*, at the angles, form the parallelograms *pq* and *tv*, by drawing *qc* parallel to BC, and *cp* to CD, and *te* to EF, and *ev* to DE.—Lastly, take B*n* = *p*C, and make the parallelogram *mn*, by drawing *nb* parallel to AB, and *bm* parallel to BC. And so on through the whole.

PROP. II.

If any number of lines, that are connected together and moveable about the points of connection, be kept in equilibrio by weights laid on the angles, as in the last proposition: Then will the weight on any angle C *be universally proportional to* $\frac{\textit{sine of the} \angle \text{BCD}}{s. \angle \text{BC}c \times s. \angle c\text{CD}}$; *that is, directly as the sine of that angle, and reciprocally as the sines of the two parts or angles into which that angle is divided by a line drawn through it perpendicular to the horizon.* See the former figure.

Demonstration.—By the last proposition the weights are as Bb, Cc, Dd, &c, where Bn = pC, Cq = rD, Ds = tE, &c. But, since the angle ABb is = the angle Bbn, and the angle BCc = the angle Ccq, &c, these being always the alternate angles made by a line cutting two other parallel lines; also the sine of the ∠ ABC = s. ∠ Bnb, and s. ∠ BCD = s. ∠ Cqc, these being supplements to each other; by plane trigonometry we shall have,

$$(\text{B}n =)\ \frac{\text{B}b \times s. \angle \text{AB}b}{s. \angle \text{ABC}} = (\text{C}p =)\ \frac{\text{C}c \times s. \angle c\text{CD}}{s. \angle \text{BCD}},$$

$$(\text{C}q =)\ \frac{\text{C}c \times s. \angle \text{BC}c}{s. \angle \text{BCD}} = (\text{D}r =)\ \frac{\text{D}d \times s. \angle d\text{DE}}{s. \angle \text{CDE}},$$

$$(\text{D}s =)\ \frac{\text{D}d \times s\ \angle \text{CD}d}{s. \angle \text{CDE}} = (\text{E}t =)\ \frac{\text{E}e \times s. \angle e\text{EF}}{s. \angle \text{DEF}},$$

and so on. Hence,

$$\text{B}b : \text{C}c :: \frac{s. \angle \text{ABC}}{s. \angle \text{AB}b} : \frac{s. \angle \text{BCD}}{s. \angle c\text{CD}},$$

$$\text{C}c : \text{D}d :: \frac{s. \angle \text{BCD}}{s. \angle \text{BC}c} : \frac{s. \angle \text{CDE}}{s. \angle d\text{BE}},$$

$$\text{D}d : \text{E}e :: \frac{s. \angle \text{CDE}}{s. \angle \text{CD}d} : \frac{s. \angle \text{DEF}}{s. \angle e\text{EF}}, \text{ \&c.}$$

Or, by dividing the latter terms of the first of these proportions each by s. ∠ bBC, and then compounding together two of the proportions, then three of them, &c, striking out the common factors, and observing that the s. ∠ bBC is =

s. ∠ BCc, the s. ∠ cCD = s. CDd, &c, we shall have the following proportions; viz,

$$Bb : Cc :: \frac{s.\angle ABC}{s.\angle ABb \times s.\angle bBC} : \frac{s.\angle BCD}{s.\angle BCc \times s.\angle cCD},$$

$$Bb : Dd :: \frac{s.\angle ABC}{s.\angle ABb \times s.\angle bBC} : \frac{s.\angle CDE}{s.\angle CDd \times s.\angle dDE},$$

$$Bb : Ee :: \frac{s.\angle ABC}{s.\angle ABb \times s.\angle bBC} : \frac{s.\angle DEF}{s.\angle DEe \times s.\angle eEF},$$

and so on.

Otherwise.

Since Cp or Bn : Bm or nb :: s. ∠ Bbn,

or s. ∠ ABb : s. ∠ bBC or s. ∠ BCc :: $\frac{1}{s.\angle BCc} : \frac{1}{s.\angle ABb}$;

and Cp or qc : Cq or Dr :: s. ∠ cCq or s. ∠ CDd : s. ∠ cCq or

s. ∠ BCc :: $\frac{1}{s.\angle BCc} : \frac{1}{s.\angle CDd}$;

it is clear that Cp is as $\frac{1}{s.\angle BCc}$; that is, the forces mB, pC, rD, &c. are always reciprocally as the sines of the angles which they make with the vertical line.

And since Cc = $\frac{Cp \times s.\angle Cpc}{s.\angle cCp} = \frac{Cp \times s.\angle BCD}{s.\angle cCD}$;

therefore any force or weight Cc is as $\frac{s.\angle BCD}{s.\angle cCB \times s.\angle cCd}$.

And this is the same as the property in corol. 4 to the 3d proposition following.

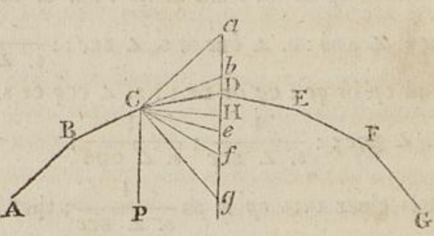
Corol. If DC be produced to h; then, the sine of the angle hCB being equal to the sine of its supplement BCD, the same weight or force Cc will be always proportional to $\frac{s.\angle hCB}{s.\angle BCc \times s.\angle DCc}$; which three angles together make up two right angles.

Properties similar to the foregoing are otherwise determined in the following propositions.

PROP. III.

Let there be any number of lines, or bars, or beams, AB, BC, CD, DE, *&c. all in the same vertical plane, connected together and freely moveable about the joints or angles* A, B, C, D, E, *&c, and kept in equilibrio by their own weights, or by weights only laid on the angles: It is required to assign the proportion of those weights; as also the force or push in the direction of the said lines; and the horizontal thrust at every angle.*

Solution.—Through any point, as D, draw a vertical line *a*DH*g*, &c: to which, from any point, as C, draw lines in the direction



of, or parallel to, the given lines or beams, viz, C*a* parallel to AB, and C*b* parallel to BC, and C*e* to DE, and C*f* to EF, and C*g* to FG, &c; also CH parallel to the horizon, or perpendicular to the vertical line *a*D*g*, in which also all these parallels terminate.

Then will all those lines be exactly proportional to the forces acting or exerted in the directions to which they are parallel, and of all the three kinds, viz. vertical, horizontal, and oblique. That is, the oblique forces or thrusts in direction of the bars . . . . AB, BC, CD, DE, EF, FG, are proportional to their parallels . . C*a*, C*b*, CD, C*e*, C*f*, C*g*; and the vertical weights on the angles B, C, D, E, F, &c, are as the parts of the vertical . . . . *ab*, *b*D, D*e*, *ef*, *fg*, and the weight of the whole frame ABCDEFG, . . . is proportional to the sum of all the verticals, or to *ag*; also the horizontal thrust, at every angle, is every where the same constant quantity, and is expressed by the constant ho-

Demonstration.—All these proportions of the forces derive and follow immediately from the general well known property, in Statics, that when any forces balance and keep each other in equilibrio, they are respectively in proportion as the lines drawn parallel to their directions, and terminating each other.

Thus, the point or angle B is kept in equilibrio by three forces, viz, the weight laid and acting vertically downward on that point, and by the two oblique forces or thrusts of the two beams AB, CB, and in these directions. But C*a* is parallel to AB, and C*b* to BC, and *ab* to the vertical weight; those three forces are therefore proportional to the three lines *ab*, C*a*, C*b*.

In like manner, the angle C is kept in its position by the weight laid and acting vertically on it, and by the two oblique forces or thrusts in the direction of the bars BC, CD: consequently these three forces are proportional to the three lines *b*D, C*b*, CD, which are parallel to them.

Also, the three forces keeping the point D in its position, are proportional to their three parallel lines D*e*, CD, C*e*.—And the three forces balancing the angle E, are proportional to their three parallel lines *ef*, C*e*, C*f*.—And the three forces balancing the angle F, are proportional to their three parallel lines *fg*, C*f*, C*g*. And so on continually, the oblique forces or thrusts in the directions of the bars or beams, being always proportional to the parts of the lines parallel to them, intercepted by the common vertical line; while the vertical forces or weights, acting or laid on the angles, are proportional to the parts of this vertical line intercepted by the two lines parallel to the lines of the corresponding angles.

Again, with regard to the horizontal force or thrust: since the line DC represents, or is proportional to the force in the direction DC, arising from the weight or pressure on the angle D; and since the oblique force DC is equivalent to, and resolves into, the two DH, HC, and in those directions, by the resolution of forces, viz, the vertical force DH, and the horizontal force HC; it follows, that the horizontal force or thrust at the angle D, is proportional to the line CH; and the part of the vertical force or weight on the angle D, which produces the oblique force DC, is proportional to the part of the vertical line DH.

In like manner, the oblique force *cb*, acting at C, in the direction CB, resolves into the two *b*H, HC; therefore the horizontal force or thrust at the angle C, is expressed by the line CH, the very same as it was before for the angle D; and the vertical pressure at C, arising from the weights on both D and C, is denoted by the vertical line *b*H.

Also, the oblique force *a*C, acting at the angle B, in the direction BA, resolves into the two *a*H, HC; therefore again the horizontal thrust at the angle B, is represented by the line CH, the very same as it was at the points C and D; and the vertical pressure at B, arising from the weights on B, C, and D, is expressed by the part of the vertical line *a*H.

Thus also, the oblique force C*e*, in direction DE, resolves into the two CH, H*e*, being the same horizontal force with the vertical H*e*; and the oblique force C*f*, in direction EF, resolves into the two CH, H*f*; and the oblique force C*g*, in direction FG, resolves into the two CH, H*g*; and the oblique force C*g*, in direction FG, resolves into the two CH, H*g*; and so on continually, the horizontal force at every point being expressed by the same constant line CH; and the vertical pressures on the angles by the parts of the verticals, viz, *a*H the whole vertical pressure at B, from the weights on the angles B, C, D; and *b*H the whole pressure on C from the weights on C and D; and DH the part of the weight on D causing the oblique force DC; and H*e* the other part of the weight on D causing the oblique pressure DE; and H*f* the whole vertical pressure at E from the weights on D and E; and H*g* the whole vertical pressure on F arising from the weights laid on D, E and F. And so on.

So that, on the whole,
*a*H denotes the whole weight on the points from D to A;
and H*g* the whole weight on the points from D to G;
and *ag* the whole weight on all the points on both sides;

while *ab*, *b*D, D*e*, *ef*, *fg* express the several particular weights laid on the angles B, C, D, E, F.

Also, the horizontal thrust is every where the same constant quantity, and is denoted by the line CH.

Lastly, the several oblique forces or thrusts, in the directions AB, BC, CD, DE, EF, FG, are expressed by, or are proportional to, their corresponding parallel lines, C*a*, C*b*, CD, C*e*, C*f*, C*g*.

Corollary 1. It is obvious, and remarkable, that the lengths of the bars AB, BC, &c, do not affect or alter the proportions of any of these loads or thrusts; since all the lines C*a*, C*b*, *ab*, &c, remain the same, whatever be the lengths of AB, BC, &c. The positions of the bars, and the weights on the angles depending mutually on each other, as well as the horizontal and oblique thrusts. Thus, if there be given the position of DC, and the weights or loads laid on the angles D, C, B; set these on the vertical, DH, D*b*, *ba*, then C*b*, C*a* give the directions or positions of CB, BA, as well as the quantity or proportion CH of the constant horizontal thrust.

Corol. 2. If CH be made radius; then it is visible that H*a* is the tangent, and C*a* the secant of the elevation of C*a* or AB above the horizon; also H*b* is the tangent and C*b* the secant of the elevation of C*b* or CB; also HD and CD the tangent and secant of the elevation of CD; also H*e* and C*e* the tangent and secant of the elevation of C*e* or DE; also H*f* and C*f* the tangent and secant of the elevation of EF; and so on; also the parts of the vertical *ab*, *b*D, *ef*, *fg*, denoting the weights laid on the several angles, are the differences of the said tangents of elevations. Hence then in general,

1st. The oblique thrusts, in the directions of the bars, are to one another, directly in proportion as the secants of their angles of elevation above the horizontal directions; or, which is the same thing, reciprocally proportional to the cosines of the same elevations, or reciprocally proportional to the sines of the vertical angles, a, b, D, e, f, g, &c, made by the vertical line with the several directions of the bars; because the secants of any angles are always reciprocally in proportion as their cosines.

2. The weight or load laid on each angle, is directly proportional to the difference between the tangents of the elevations above the horizon, of the two lines which form the angle.

3. The horizontal thrust at every angle, is the same constant quantity, and has the same proportion to the weight on the top of the uppermost bar, as radius has to the tangent of the elevation of that bar. Or, as the whole vertical ag, is to the line CH, so is the weight of the whole assemblage of bars, to the horizontal thrust. Other properties also, concerning the weights and the thrusts, might be pointed out, but they are less simple and elegant, than the above, and are therefore omitted; the following only excepted, which are inserted here on account of their usefulness.

Corollary 3. It may hence be deduced also, that the weight or pressure laid on any angle, is directly proportional to the continual product of the sine of that angle and of the secants of the elevations of the bars or lines which form it. Thus, in the triangle bCD, in which the side bD is proportional to the weight laid on the angle C, because the sides of any triangle are to one another as the sines of their opposite angles, therefore as sin. D : Cb : : sin. bCD : bD; that is, bD is as $\frac{\text{sin. } b\text{CD}}{\text{sin. D}} \times \text{C}b$; but the sine of angle D is the cosine of the elevation DCH, and the cosine of any angle is reciprocally proportional to the secant, therefore bD is as sin. bCD × sec. DCH × Cb; and Cb being as the secant of the angle bCH of the elevation of bC or BC above the horizon, therefore bD is as sin. bCD × sec. bCH × sec. DCH; and the sine of bCD being the same as the sine of its supplement BCD; therefore the weight on the angle C, which is as bD, is as the sin. BCD

× sec. DCH × sec. *b*CH, that is, as the continual product of the sine of that angle and the secants of the elevations of its two sides above the horizon.

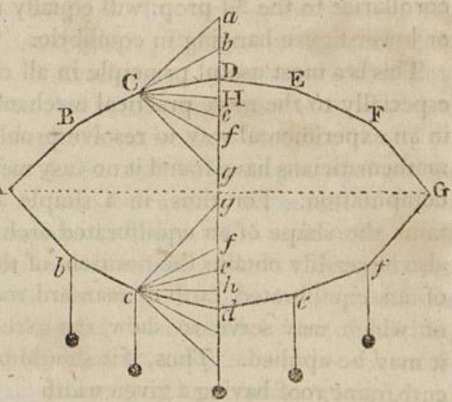
Corol. 4.—Further, it easily appears also, that the same weight on any angle C, is directly proportional to the sine of that angle BCD, and inversely proportional to the sines of the two parts BCP, DCP, into which the same angle is divided by the vertical line CP. For the secants of angles are reciprocally proportional to their cosines or sines of their complements: but BCP = C*b*H, is the complement of the elevation *b*CH, and DCP is the complement of the elevation DCH; therefore the secant of *b*CH × secant of DCH is reciprocally as the sin. BCP × sin. DCP; also the sine of *b*CD is = the sine of its supplement BCD; consequently the weight on the angle C, which is proportional to sin. *b*CD × sec. *b*CH × sec. DCH, is also proportional to $\frac{\text{sin. BCD}}{\text{sin. BCP} \times \text{sin. DCP}}$, when the whole frame or series of angles is balanced, or kept in equilibrio, by the weights on the angles; the same as in the preceding proposition.

Scholium.—The foregoing proposition is very fruitful in its practical consequences, and contains the whole theory of arches, which may be deduced from the premises by supposing the constituting bars to become very short, like arch stones, so as to form the curve of an arch. It appears too, that the horizontal thrust, which is constant or uniformly the same throughout, is a proper measuring unit, by means of which to estimate the other thrusts and pressures by, as they are all determinable from it and the given positions; and the value of it, as appears above, may be easily computed from the uppermost or vertical part alone, or from the whole assemblage together, or from any part of the whole, counted from the top downwards.

The solution of the foregoing proposition depends on this consideration, viz, that an assemblage of bars or beams, being connected together by joints at their extremities, and freely moveable about them, may be placed in such a vertical position, as to be exactly balanced, or kept in equilibrio, by their mutual thrusts and pressures at the joints; and that the effect will be the same if the bars themselves be considered as without weight, and the angles be pressed down by laying on them weights which shall be equal to the vertical pressures at the same angles, produced by the bars in the case when they are considered as endued with their own natural weights. And as we have found that the bars may be of any length, without affecting the general properties and proportions of the thrusts and pressures, therefore by supposing them to become short, like arch stones, it is plain that we shall then have the same principles and properties accommodated to a real arch of equilibration, or one that supports itself in a perfect balance. It may be further observed, that the conclusions here derived, in this proposition and its corollaries, exactly agree with those derived in a very different way, in the former editions of the principles of bridges, viz, in props. 1 and 2, and their corollaries; and which have been here repeated, in the foregoing prop. 2.

PROP. IV.

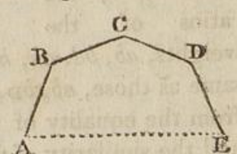
If the whole figure in the third proposition be inverted, or turned round the horizontal line AG *as an axis, till it be completely reversed, or in the same vertical plane below the first position, each angle* D, *d*, &c, *being in the same plumb line; and if weights i, k, l, m, n, which are respectively equal to the weights laid on the angles* B, C, D, E, F, *of the first figure, be now suspended by threads from the corresponding angles b, c, d, e, f, of the lower figure; then will those weights keep this figure in exact equilibrio, the same as the former, and all the tensions or forces in the latter case, whether vertical or horizontal or oblique, will be exactly equal to the corresponding forces of weight or pressure or thrust in the like directions of the first figure.*

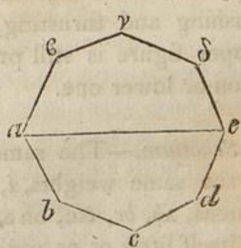


This necessarily happens, from the equality of the weights, and the similarity of the positions and actions of the whole in both cases. Thus, from the equality of the corresponding weights, at the like angles, the ratios of the weights, *ab*, *bd*, *dh*, *he*, &c, in the lower figure, are the very same as those, *ab*, *b*D, DH, H*e*, &c, in the upper figure; and from the equality of the constant horizontal forces CH, *ch*, and the similarity of the positions, the corresponding vertical lines, denoting the weights, are equal, namely, *ab* = *ab*, *b*D = *bd*, *dh* = DH, &c. The same may be said of the oblique lines also, *ca*, *cb*, &c, which being parallel to the beams A*b*, *bc*, &c, will denote the tensions of these, in the direction of their length, the same as the oblique thrusts or pushes in the upper figures. Thus, all the corresponding weights and actions, and positions, in the two situations, being exactly equal and similar, changing only drawing and tension for pushing and thrusting, the balance and equilibrium of the upper figure is still preserved the same in the hanging festoon or lower one.

Scholium.—The same figure, it is evident, will also arise, if the same weights, *i*, *k*, *l*, *m*, *n*, be suspended at like distances, A*b*, *bc*, &c, on a thread, or cord, or chain, &c, having in itself little or no weight. For the equality of the weights, and their directions and distances, will put the whole line, when they come to equilibrium, into the same festoon shape of figure. So that, whatever properties are inferred in the corollaries to the 3d prop. will equally apply to the festoon or lower figure hanging in equilibrio.

This is a most useful principle in all cases of equilibriums, especially to the mere practical mechanist, and enables him in an experimental way to resolve problems, which the best mathematicians have found it no easy matter to effect by mere computation. For thus, in a simple and easy way he obtains the shape of an equilibrated arch or bridge; and thus also he readily obtains the positions of the rafters in the frame of an equilibrated curb or mansard roof; a single instance of which may serve to show the extent and uses to which it may be applied. Thus, if it should be required to make a curb frame roof having a given width AE, and consisting of four rafters AB, BC, CD, DE, which shall either be equal or in any given proportion to each other. There can be no doubt but that the best form of the roof will be that which puts all its parts in equilibrio, so that there may be no unbalanced parts, which may require the aid of ties or stays, to keep the frame in its position. Here the mechanic has nothing to do, but to take four like but small pieces, that are either equal or in the same given proportions as those proposed, and connect them loosely together at the joints A, B, C, D, E, by pins or strings, so as to be freely moveable about them; then suspend this from two pins, *a*, *e*, fixed in a horizontal line, and the chain of the pieces will arrange itself in such a festoon or form, *abcde*, that all its parts will come to rest in equilibrio. Then, by inverting the figure, it will exhibit the form and frame of a curb roof $a\beta\gamma\delta e$, which will also be in equilibrio, the thrusts of the pieces now balancing each other, in the same manner as was done by the mutual pulls or tensions



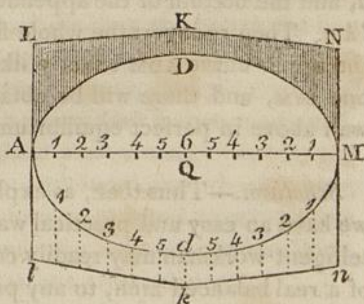


of the hanging festoon *a b c d e*. By varying the distance *ae*, of the points of suspension, moving them nearer to, or farther off, the chain will take different forms; then the frame ABCDE may be made similar to that form which has the most pleasing or convenient shape, found above as a model.

Indeed this principle is very fruitful in its practical consequences. It is easy to perceive that it contains the whole theory of the construction of arches: for each stone of an arch may be considered as one of the rafters or beams in the foregoing frames, since the whole is sustained by the mere principle of equilibration, and the method, in its application, will afford some elegant and simple solutions of the most difficult cases of this important problem; some examples of which will be shown hereafter.

PROP. V.

To form mechanically a balanced Festoon arch, on the principles of the last proposition; having a given pitch or height and span, and also a given height and form of wall or roadway over it.



Let AM be the given or proposed span of the arch, DQ its pitch or greatest height, DK the thickness at the crown, and ALKNM the given anterior form of the wall: in order to determine the form of the curve ADM which shall put that wall in equilibrio.

Invert the whole figure ALKNM, as in the opposite position A*lkn*M, or construct this latter figure, on the lower side of AM, exactly equal and similar to the proposed upper one; the point *d* answering to the point D, and the point *k* to the point K, &c. Let a very fine and thin, but strong line, such as a fine silken cord, or a bricklayer's working line, or perhaps a very fine and slender chain of small links, be suspended from the extreme points A and M, and of such a length, that its middle point may hang at the point *d*, or a little below it. Divide the given span or width AM into a number of equal parts, the more the better, as at the points 1, 2, 3, 4, 5, &c; from which draw vertical lines, cutting the festoon chain or cord in the corresponding points 1, 2, 3, 4, 5, &c. Then take short pieces of another chain, and suspend them by these points of the festoon 1, 2, 3, &c, as represented by the dotted verticals in the lower part of the figure. This will somewhat alter the form of the curve. If now the new curve should correspond with the point *d*, and all the bottoms of the vertical pieces of appended chain also coincide with the given line of roadway *l k n*, the business is done. But if both those coincidences do not take place, then alterations must be made, by trials and by judgment, in lengthening or shortening either the festoon A*d*M, or the appended vertical pieces of chain, or in both, till such time as those coincidences are accomplished, namely, the bottom of the arch with the point *d*, and the bottom of the appended pieces with the boundary *l k n*. Then re-invert the whole figure, or otherwise trace out the upper curve ADM exactly like or the same as the lower one A*d*M, and there will be obtained an arch sustaining the wall above in perfect equilibrium.

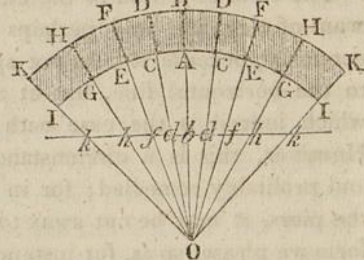
Scholium.—Thus then, as explained by professor Robison, we have an easy and practical way, by which any common intelligent workman may readily construct for himself the form of a real balanced arch, to any proposed design for a bridge. In this method, the thinner and lighter the festoon line is, so as to bear but a small proportion to the weight of the appended pieces of chain, so much the more exact will the conclusion be obtained, when the superincumbent wall is of uniform weight of masonry. But as the festoon line represents the line of voussoirs or arch stones, in the constructed arch, if these only are solid, and the rest of the wall or matter above them be looser and lighter, then there ought to be an equality of proportion between the weights of the festoon chain and the string or rib of arch stones, and between the superior wall and the appended pieces of chain; a circumstance of equality to be obtained by mutual accommodations and calculations adapted to the real circumstances of the case.

The chief objection to the curve found in this way is a want of elegance, and perhaps too of convenience and of economy, because it does not spring or rise at right angles to the horizontal line, but at a much smaller angle; and which indeed is the case with all curves of equilibration. However, this is a circumstance which can be very safely and profitably remedied; for in the part of the flanks near the piers, it may be cut away to hollow the arch out to any form we please, so as, for instance, to resemble the elliptical arch, which is one of the most graceful of all; because the masonry is so solid and strong in that part. And this will be not only more agreeable to the eye, but will also leave more room for water and boats to pass, and will be a saving in the expence of masonry. To accomplish this end with more regularity and method, instead of dividing the horizontal line into equal parts at the points 1, 2, 3, &c, if the festoon chain itself be so divided, viz, into equal parts, and the pieces of chain be appended at these, in the manner before mentioned, then the greater number of these pieces being thus near the extremities, they will draw the arch more down in that part, and thus hollow it out there in a more regular and uniform manner, making the shape more pleasing and commodious, and yet leaving it sufficiently near a true balance.

The following proposition is here added, to determine the figure of a balanced arch, on the supposition that the voussoirs are at liberty to slide on each other. A principle indeed having no real foundation in fact, though it has been much insisted on by some persons.

PROP. VI.

It is proposed to determine the nature and properties of a balanced arch, as derived from the property of the wedge, or by considering the voussoirs or arch-stones as frustrums of wedges.



Let ACEGI &c, be the inner or lower curve of an arch, formed of the voussoirs, or wedge pieces, the vertical sections of which are the quadrilaterals AD, CF, EH, GK, &c, considered as so many elementary parts of the arch, the upper sides of them forming the exterior or outer curve BDFHK, and their butting sides making the joints AB, CD, EF, GH, IK, &c, which joints produced, meet in the point O, of the vertical line OAB. Through any point *b*, in that line, draw the horizontal line *bdfhk*, or perpendicular to the vertical line OAB, and cutting the directions of the joints in the respective or corresponding points *b*, *d*, *f*, *h*, *k*, &c.

Now every wedge in the balanced arch, supposing its sides polished, must be kept in equilibrio, in its place, by the mutual action of three forces, viz, by its own weight acting in a direction perpendicular to the horizon, and by the thrust or pressure of the two adjacent wedges, one on each side, in directions perpendicular to their sides, or to the joints: So, for instance, the wedge AD is balanced, or kept in equilibrio, by its own weight acting in the vertical direction BO, and by two forces acting perpendicularly to AB and CD; and the stone CF, by its weight in the vertical direction, and by two forces perpendicular to CD and EF; also the stone EH, by its weight acting vertically, and by two forces perpendicular to EF and GH; also the stone GK, by its weight vertically, and by two forces perpendicular to GH and IK ; and so on, the weights all acting in the vertical direction parallel to BAO.

But, whenever three forces balance one another, they have then to each other the same ratios as the sides of a triangle drawn perpendicular to the directions of the forces. Therefore the three forces balancing the wedge AD, are proportional to the three sides of the triangle *obd*, these sides being respectively perpendicular to those forces, viz, the side *bd* perpendicular to the vertical direction of gravity, also *ob* perpendicular to the force against the joint AB, and *od* perpendicular to the force against the joint CD. For the same reason the wedge CF is balanced by three forces proportional to the three sides *df*, *od*, *of*, of the triangle *odf*; and the wedge EH by forces proportional to the three sides *fh*, *of*, *oh*, of the triangle *ofh* ; and the wedge GK by forces proportional to the three sides *hk*, *oh*, *ok*, of the triangle *ohk* ; and so on. So that, in all these cases, the weights of the wedges, and their oblique push perpendicular to the joints, will have these following ratios, viz,

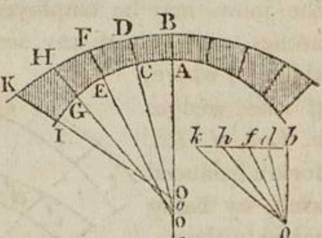
the weights of the wedges - - - AD, CF, EH, GK, &c,
as the parts of the horizontal - - *bd*, *df*, *fh*, *hk*, &c,
and the push at the joints as - - *ob*, *od*, *of*, *oh*, &c,
also the sums of the wedges, or the parts, AD, AF, AH, AK,
are proportional to the perpendiculars *bd*, *bf*, *bh*, *bk*,
which are the tangents of the angles BOD, BOF, BOH, BOK, &c, of which the oblique thrusts *od*, *of*, *oh*, *ok*, are the secants, to the radius *ob*, which denotes the constant push in the horizontal direction at every wedge, or every point of the arch. Which, on the whole, amounts to this, viz, that the weights of any part of the balanced arch, or set of wedges, commencing from the vertex, are directly proportional to the tangents of the angles which the joints make with the vertical line or direction, while the oblique thrusts, in the directions of the arch at the extremity, or perpendicular to the joints, are proportional to the secants of the same angles ; the constant horizontal push, at every point, being proportional to the radius.

And this property comes to the very same thing as the properties in the foregoing propositions, because the angles of elevation of the curve at every point, or of the direction of the tangents there, or of the curve itself, are equal to the angles in this proposition, which the joints form with the vertical direction. So that, all the three theories in these four propositions are all one and the same in effect, amounting to the very same thing, and yielding the same conclusions. And therefore, whatever consequences may further be drawn from any one of them, may be understood as deduced from the whole.

Scholium.—In the practice of bridge-building, the key piece, or wedge at the crown, is a solid, having its magnitude and weight half on each side of the middle vertical line; whereas, in this proposition, it has been supposed that this wedge is divided and actually separated in two by that line AB: this however will cause no difference in the theory, nor yet in the practice; for, in any calculations that may be required, it is only necessary to suppose the key piece divided exactly in the middle, then taking half its weight for the weight of the piece AD, and computing all the other weights and angles from the middle line AB.

It has also been supposed, in all the three theories that have been contemplated, that the constituent parts are formed of materials perfectly smooth and polished, and put together without cement, and without all kinds of ties or bars, so as to leave them quite at liberty to slide over each other, the parts being kept in a perfect balance by means of their shape, weight, and disposition only. This, it must be acknowledged, is not the case in real practice; as here all the materials are quite rough, which very much prevents them from sliding by each other, even when their abutting surfaces are laid at a considerable slope or angle. But this circumstance however, so far from being a disadvantage, by thus deviating from the theory, is on that very account of great use and benefit. For, the equilibrium among the constituent parts of the arch, established by the foregoing theories, is of that nice and critical nature, that the whole hangs in a kind of tottering state of balance, from the perfect polish of the parts, so that any the least accidental extraneous force or pressure, on any particular part, would destroy the equilibrium, and cause the whole to fall down, except for the length of the joints and stones. The theory also supposes the parts, constituting the fabric, to be exceedingly small, and may be even round, small, polished globules. But because of the shape and roughness and magnitude of the parts, of which an arch is constituted, it comes to pass, that a moderate degree of imperfection in the structure, or any accidental shocks or pressure from external objects, has no sensible effect in displacing or deranging the materials: for the wedge-like form prevents any piece from easily dropping out by itself; and the roughness of the sides prevents the wedges from sliding; also the considerable magnitude of the stones, or other matter, while it enables them to bear the weight and pressure of the whole fabric, without being crushed to pieces, admits of a small displacing of materials, or deviation from a perfect balance, as prescribed by theory, without suffering any sensible inconvenience.

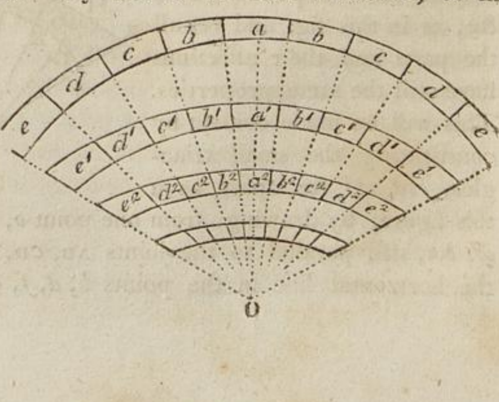
It has been supposed in this proposition, that the directions of the joints, CD, EF, GH, &c, when produced, all meet in the same point O, of the vertical line OAB. This however is not necessary in the theory; as the directions of the joints may meet the vertical in so many different points O, O, O, &c, as in this fig. and yet all the parts and their affections have still the same properties. This will be made evident by constituting the small triangles, *obd*, *obf*, &c, apart, as in this figure, by drawing, from one point *o*, the lines *ob*, *od*, *of*, &c, still parallel to the joints AB, CD, EF, &c, meeting the horizontal line in the points *b*, *d*, *f*, &c: for, because



these lines are perpendicular to the actions of the forces, of pressure and push, of the arch pieces, the same proportions among these, as before deduced, still take place, and hold good; viz. that the weights are in proportion as the parts of the line *bdfhk*, and the oblique push as the corresponding lines *ob*, *od*, *of*, &c, of which *ob* is as the horizontal thrust.

It has also been supposed, that the joints are cut or drawn perpendicular to the inner curve at every point, or that all the angles at it, C, E, &c, are right-angles. But neither is this necessary in the theory; for the system of balancing will be still the same, whatever those angles may be, whether all alike or all various, as these differences will only cause an alteration in the weight or length of the arch-pieces, which still will be represented in their proportions by the parts of the line *bdfhk*. And indeed we often see this kind of oblique joints employed in the small arches in the common practice of architecture and building, as over windows, doors, gateways, &c. But yet such a practice is not to be admitted into the larger kind of arches, employed in bridges, &c, as being both ungraceful and troublesome, as well as weakening the fabrick.

It is manifest, from all the theories, that the balancing of the arch is not restricted to any particular kind of curve or shape, for either the under or upper curve; as the arch may be balanced with any particular curves we please. It also follows very evidently, that the same angles or directions of the joints may be employed to balance a great variety of arches, and indeed any sort of an arch whatever; as in this fig.; where, if the wedges *a*, *b*, *c*, *d*, &c, form a balanced arch, by being taken in the required proportion to each other, viz, as the differences of the



tangents of the angles formed by their sides with the vertical line; then, if the under curve of any of the other lower arches be assumed of any shape at pleasure, the upper curve of them will be found, by taking their corresponding wedges, *a*1, *b*1, *c*1, &c, or *a*2, *b*2, *c*2, &c, or *a*3, *b*3, *c*3, &c, in the same proportions to each other as the wedges *a*, *b*, *c*, &c, are in the uppermost arch; and all the sets of wedges will form balanced arches.

EXAMPLE.

The theory laid down in the preceding propositions, which give, all of them, the same conclusions, will serve as a foundation on which to establish a method for constructing arches of equilibration, on any proposed curve whatever. The method however will require some further preparation, to render the application to practice easy and convenient. We may here, however, in the mean time, just take one example, in order to show the facility of the mode of calculation from the theory, so far as it has now been laid down. In this example, we shall suppose that the intrados curve is a circular arc, which is formed by the under sides of the wedge pieces, the joints between which are all perpendicular to that curve, as the only proper position, or all directed exactly to the centre of the curve. We shall also suppose the wedge pieces to form equal parts of that arc, of the quantity of 5° each, that is, each wedge subtending at the centre an angle of 5 degrees, the key, or middle wedge at the crown, therefore, extending 2 degrees and a half on each side of the vertical line passing through the centre; and have 17 other wedges, of equal angle (5°) on each side of the key, making in all 35 wedges, which, at 5 degrees each, will form an entire arch of 175 degrees. In this case, the angle which the sides of the middle wedge forms with the middle vertical line, will be that of half the breadth of the wedge, or $2\frac{1}{2}$ degrees; and the angles which the sides of the other wedges, on each hand of the crown or key wedge, form with the vertical direction, will be found by adding continually the breadth of each wedge (5 degrees), to the said $2\frac{1}{2}$ degrees; by which it will be found that the angles at the centre, formed with the vertical, by the said lower edges of the arch pieces, in order after the key, will be as follows, viz, that of the 2d wedge $7\frac{1}{2}$ degrees; that of the 3d, $12\frac{1}{2}$ degrees; that of the 4th, $17\frac{1}{2}$ degrees; and so on to the 17th or last on each side the key, which will have its lower edge making an angle of $87\frac{1}{2}$ degrees with the vertical direction: all which angles, of inclination to the vertical, are ranged in the 2d column of the following tablet, the first, or half the middle wedge, making an angle of $2\frac{1}{2}$ degrees. We shall also suppose the weight of the middle wedge at the crown to be a certain given quantity, represented by unity or 1, and express the several other weights and pressures, as in the other columns of the said tablet, in terms of that unit: so that all these proportional numbers for the other weights and pressures, will require to be multiplied by any other weight of middle wedge which may happen to occur in any other case.

Now, in regard to the rule for computing all the other weights and pressures, according to the conclusions from the preceding theory, it is very easy and simple indeed, viz, that the weight of any part of the arch, counted from the vertex or crown downward, is always proportional to the tangent of the angle of inclination of the lower wedge to the vertical, while the oblique push or pressure, in direction of the curve, is proportional to the secant of the same angle, and the constant horizontal thrust is proportional to the radius. For which reason it is, as formerly observed, that the constant horizontal thrust is a proper radical measuring unit, by means of which to compute the two other circumstances, namely, the weight of the arch, and the oblique push or pressure in the direction of the curve: for, the horizontal thrust being taken for radius, then the weight of the semiarch will be the tangent of the angle with the vertex, and the oblique pressure the secant of the same angle, to that radius. Consequently, if the constant horizontal push be called h, then the weight of the semiarch will be $h \times t$, or h

multiplied by the tangent of the side's inclination to the vertical, and the oblique pressure of the arch will be $h \times s$, or h multiplied by the secant of the same angle. So that, in calculating the said several weights and oblique pushes of the arches, we have nothing to do but to take out, from a trigonometrical table, the tangents and secants of the several angles of inclination to the vertical, as contained in the 2d column of the tablet, and multiply all the tangents and secants by the number expressing the constant horizontal thrust, for all the values of the several weights and pressures, as arranged in the 3d and 4th columns of the tablet; the products of the tangents being the several weights of the half arches, in the 4th column, and the products of the secants being the oblique pressures of the same in the arch's direction, as in the 3d column. This calculation will be rendered still easier by using the log. tangents and secants; for there will then be nothing to do, but to take out all the log. tangents and secants; then to each of them add the constant log. of the horizontal thrust; lastly, take out the natural numbers answering to these sums, and they will be the required weights and pressures.

As to the uniform horizontal thrust, which is the constant multiplier, its value is easily found thus: It has been shown that this horizontal thrust is every where in the same proportion to the weight of half the middle or key wedge, as radius is to the tangent of half the angle of that wedge; that is, as $t : 1 :: \frac{1}{2}w : \frac{1}{2}w \div t = h$ the horizontal thrust, putting w for the weight of the key piece, and t for the tangent of half its angle; or, if we put its weight $w = 1$, then this will become $\frac{1}{2} \div t = h$ the horizontal thrust. Now, in the example, the angle subtended by the key is 5 degrees, the half of which is $2\frac{1}{2}$ degrees, and the tangent of this is $\cdot 0436609$; then $\frac{1}{2}$ or $\cdot 5 \div \cdot 0436609 = 11\cdot 451883 = h$ the constant horizontal thrust, that is, 11 times the weight of the key piece and nearly one half more; or, the same may be easier found from the cotangent of the same angle $2\frac{1}{2}$ degrees, which is $22\cdot 903766$, the cotangent of any angle being equal to the reciprocal of its tangent, to the radius 1;

therefore, in general, $\frac{1}{2} \div$ tang. $= \frac{1}{2}$ the cotang. is $= h$ the horizontal thrust, and in the present instance the half of the cotangent 22·903766 is 11·451883 the same value of the horizontal thrust as before.

Hence then the constant number 11·451883 is to be multiplied by the tangents of all the vertical angles, to give the weights of the semiarch, in the 4th column, and by the secants of the same angles, to give their oblique pressures, as in the 3d column; or else, to work by the logarithms, the log. of the constant number 11·451883, which is 1·0588769, is to be added to all the log. secants and tangents of the said angles, then the corresponding natural numbers taken, and ranged in the 3d and 4th columns of the table.

The differences of the numbers in the 4th column are taken, and ranged in the 5th or last column, for the weights of the single wedge pieces taken separately, making the whole of the first or key wedge equal to 1.—The table is as follows.

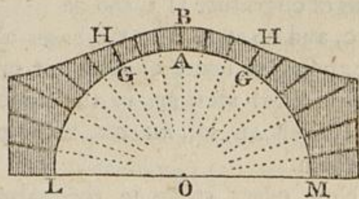
No. of sections.	Vertical angles of the joints, or $\angle$ s o.	Oblique pressures, $= h \times$ sec. $\angle$ o.	Wts. of halfarches, $= h \times$ tan. $\angle$ o.	Wts. of the sections or wedges.
	degrees.			
1	$2\frac{1}{2}$	11·46279	0·5	1·
2	$7\frac{1}{2}$	11·55070	1·50767	1·00767
3	$12\frac{1}{2}$	11·72993	2·53882	1·03115
4	$17\frac{1}{2}$	12·00763	3·61076	1·07194
5	$22\frac{1}{2}$	12·39543	4·74352	1·13276
6	$27\frac{1}{2}$	12·91065	5·96147	1·21795
7	$32\frac{1}{2}$	13·57837	7·29565	1·33418
8	$37\frac{1}{2}$	14·43478	8·78734	1·49169
9	$42\frac{1}{2}$	15·53267	10·49372	1·70638
10	$47\frac{1}{2}$	16·95094	12·49753	2·00381
11	$52\frac{1}{2}$	18·81177	14·92439	2·42686
12	$57\frac{1}{2}$	21·31377	17·97585	3·05146
13	$62\frac{1}{2}$	24·80112	21·99886	4·02301
14	$67\frac{1}{2}$	29·92521	27·64727	5·64841
15	$72\frac{1}{2}$	38·08334	36·32073	8·67346
16	$77\frac{1}{2}$	52·91028	51·65611	15·33538
17	$82\frac{1}{2}$	87·73628	86·98568	35·32957
18	$87\frac{1}{2}$	262·54113	262·29125	175·30557

From this calculation, as well as from the theorems by which it is made, it is manifest how greatly the weight and the pressure of the semiarch increase towards the bottom or the extremity, where the position of the joint approaches towards the horizontal direction, or the angle it makes with the vertical approaches towards a right-angle; and when that angle actually becomes a right-angle, or the joint quite horizontal, then the weight and pressure become equal and infinite, which must naturally be expected, both because the tangent and secant of the angle (being a right one) are then infinite, and also because it must require an infinite weight or pressure to balance there the constant given horizontal thrust, which is perpendicular to the former.

We may here, by the way, stop to examine a little in what manner the preceding calculation of the weights of the voussoirs may be employed to give a familiar and easy mechanical construction, that may approach very near to a true balanced arch. In order to this, we are to consider, that since the bases, or extents of the under sides, of all the voussoirs, are equal, it will thence happen that their weights will have to each other nearly the same ratios as their lengths, from the under to the upper side of them, or taken in the direction of the radius, that is perpendicular to the under curve or intrados, at least when the breadth or angle of these wedges is very small, which is the case in real practice, the approach to equality being the nearer indeed as their breadth is the smaller. And though the angle of 5 degrees, employed in the preceding calculation, be not such a small breadth as to render the equality and the construction perfect, it will yet serve to show the manner of proceeding in such a way of forming the arch, and will besides approach tolerably near to the truth.

As it is most proper that the joints between the wedges, in the arch of a bridge, should be in directions perpendicular to the under curve of the arch, we shall only exemplify the method in cases of that sort. For this purpose then, let us suppose the intrados or under curve to be divided into a number of equal parts, answering to a breadth of 5 degrees each, or such that the angle formed by every two adjacent joints, when produced, shall be an angle of 5 degrees. Let us then draw a line through the middle point of every one of these breadths, bisecting them, and in a direction perpendicular to the curve at every point. Then, by setting off, upon these lines, from the curve upwards, by a proper scale, lengths which shall have the same ratios to each other as the weights of the corresponding wedges through which these lines pass, or proportional to the numbers in the last column of the foregoing table; then will the lengths of these lines be the extent of the several voussoirs nearly, and therefore, their upper extremities or points being connected, by drawing short lines from one to another, they will limit or form the extrados, or the upper curve or side of the arch, when built of uniform materials, so as to be very nearly in equilibrio.

As it is manifest that the theorems and the calculation have no peculiar restricted reference to any particular curve for the intrados, or under side of the arch, we are therefore at liberty to assume that curve of any form at pleasure; therefore the form of it being so assumed, by then applying the numbers of the foregoing table to it, in the manner above mentioned, we shall have a balanced arch as required. And thus by assuming any different shapes of curve for the intrados, the same numbers in the table will give as many balanced arches as we please. Assuming then, for the inner curve, a semicircle, as in the next fig. having its span or diameter LM 84 feet, consequently its pitch or height OA 42 feet. We shall also assume AB the thickness of the crown or key-piece, equal to 6 feet, or the 14th part of the span, being nearly the proportion employed by good engineers. Dividing each half arc AL, AM, into 9 equal parts, of 10 degrees each, which will be sufficiently small to show the nature and form of the extrados, containing each an extent of two wedges or voussoirs; then from the centre O drawing radii through all the points of division, these, when continued, passing through the middle of every second wedge, the first OAB passing through the middle of the key-piece. Then, on these radii produced, set off, from the arc of the semicircle, AB, GH, &c, every second number in the last column of the table, when multiplied by 6, the assumed length of AB; then, drawing with the hand a curved line through the extremities of all the exterior lines, it will be the extrados required, exhibiting the form and limit of the wall built of uniform materials, above the circular soffit, so as to constitute an arch of equilibration nearly as in the annexed fig.

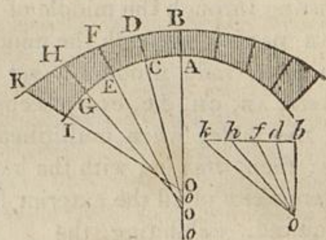


Where it is seen that the extrados follows nearly a course parallel to the intrados for about 30 degrees on each side of the vertex; after which, it begins to bend the contrary way, having there a contrary flexure during the rest of its course, going off to an infinite distance on each side parallel to the base, making the voussoirs at last of an infinite length, and composing all together a form of arch very unfit for adoption in practice.

We shall now show, in the next proposition, that, by another very strict and genuine construction, an exterior curve is derived exactly similar to the curve here obtained: in the determination of which, some part of the mode of reasoning in the demonstration of the last prop. is here again necessarily repeated.

PROP. VII.

If ACEGI *&c, be an arch, supporting a wall* ABKI, *formed of the voussoirs or arch stones* AD, CF, *&c, lying aslope, on smooth surfaces, and having the joints* AB, CD, *&c, every where perpendicular to the curve of the arch* ACE *&c. It is required to find the lengths of these arch stones, so that the whole fabric may be balanced, or kept in equilibrio.*



Let A be the vertex of the inner curve of the proposed arch; AB the given thickness of the wall at the crown, or length of the archstone there; also BAO, DCO, &c, the joints produced, making AO the radius of curvature at A, and CO at C, and EO at E, &c; the bases of the stones AC, CE, EG, GI, &c, being so many elements or small parts of the arch; and the vertical sections of the stones, or the areas of the quadrilaterals AD, CF, EH, GK, being proportional to the weights of them.

Now every stone in the balanced arch will be kept in equilibrio by three forces, viz, by its own weight acting perpendicular to the horizon, and by the pressures of the two adjacent stones, in directions perpendicular to their sides, or to the two adjacent joints: So, for instance, the stone AD is balanced, or kept in equilibrio, by its own weight, and by two forces acting perpendicularly to AB and CD; and the stone CF, by its weight, and by the two forces perpendicular to CD and EF; also the stone EH, by its weight, and by the two forces perpendicular to EF and GH; also the stone GK, by its weight, and by the two forces perpendicular to GH and IK; and so on; all these weights acting in the vertical direction BAO.

But whenever three forces balance one another, they have then the same ratios as the sides of a triangle drawn perpendicular to their directions. Therefore, if there be constructed another figure *obdfhk*, having *bk* horizontal, or perpendicular to a given vertical line *ob*; and having *od* parallel to OD, and *of* to OF, and *oh* to OH, and *ok* to OK, &c: then the three forces balancing the stone AD are proportional to the three sides of the triangle *obd*, these sides being respectively perpendicular to those forces; for the same reason, the stone CF is balanced by the three forces *df*, *od*, *of*; also the stone EH by the three *fh*, *of*, *oh*; and the stone GK by the three *hk*, *oh*, *ok*; and so on; in all these cases the weights of the stones being proportional to the bases *bd*, *df*, *fh*, *hk*, of the triangles *obd*, *odf*, *ofh*, *ohk*. But as these triangles have all the same common altitude *ob*, they have the same ratios as their bases *bd*, *df*, &c, which bases, it has been shown, are proportional to the weights of the stones, which have also been found proportional to the quadrilateral areas AD, CF, &c; therefore the quadrilaterals AD, CF, EH, GK, are respectively proportional to the triangles *obd*, *odf*, *ofh*, *ohk*.

But, as these small triangles have their angles respectively equal to the angles of the corresponding sectors, because their sides are parallel by the construction; that is, the angle *bod* = the angle BOD, &c; their areas are therefore proportional to the squares of their corresponding sides;

viz. the sectors OBD, OAC, *obd*,
proportional to OB^2, OA^2, ob^2;
and the sectors ODF, OCE, *odf*,
proportional to OD^2, OC^2, od^2; and so on.

Therefore, by taking the differences,

$$AD : obd :: OB^2 - OA^2 : ob^2,$$
$$\text{and } CF : odf :: OD^2 - OC^2 : od^2,$$
$$\text{and } EH : ofh :: OF^2 - OE^2 : of^2,$$
$$\text{and } GK : ohk :: OH^2 - OG^2 : oh^2, \text{ \&c.}$$

Hence, if ob^2 be taken $= OB^2 - OA^2$,

$$\text{then } od^2 \text{ is } = OD^2 - OC^2,$$
$$\text{and } of^2 \text{ is } = OF^2 - OE^2,$$
$$\text{and } oh^2 \text{ is } = OH^2 - OG^2, \text{ \&c.}$$

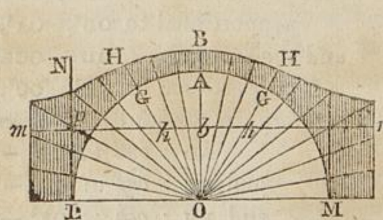
Or, by transposing, $OB^2 = OA^2 + ob^2$,

$$\text{and } OD^2 = OC^2 + od^2,$$
$$\text{and } OF^2 = OE^2 + of^2,$$
$$\text{and } OH^2 = OG^2 + oh^2, \text{ \&c.}$$

Which gives us the following geometrical construction, viz, Produce the joints till OA, OC, OE, OG, &c, be equal to the several radii of curvature at the corresponding points, A, C, E, &c; to which also draw the parallels *ob*, *od*, *of*, &c. Then take $ob = \sqrt{OB^2 - OA^2}$, and draw *bdfhk* perpendicular to ob. Lastly, make OD $= \sqrt{\text{OC}^2 + od^2}$, and OF $= \sqrt{\text{OE}^2 + of^2}$, and OH $= \sqrt{\text{OG}^2 + oh^2}$, &c; then shall the line or curve drawn through all the points B, D, F, H, K, &c, be the top of the wall, so as the whole fabric may be balanced, or kept in equilibrio, by the mutual weights and pressures of the stones, having smooth or polished sides, and at liberty to descend along them.

Note.—When the given interior curve ACE &c, is a circle, all the radii of curvature will be equal to each other, and will all have the same centre O. But in other curves, having various degrees of curvature, the radii and centres of curvature will be all different.

EXAMPLE.



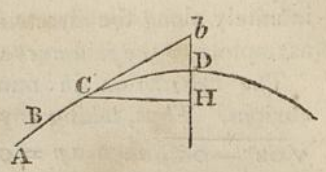
Suppose the interior curve to be a Semicircle. And suppose the span or diameter LM to be 84 feet, the height or pitch OA 42 feet, and the thickness at the crown AB 6 feet, which is the 14th part of the span. Then take ob so, that ob^2 be equal to $\text{OB}^2 - \text{OA}^2$, or $ob = \sqrt{\text{OB}^2 - \text{OA}^2}$ $= 23{\cdot}2379$, and through b draw mbn parallel to the base LM; from the centre O draw a number of radii OhGH &c, cutting the circle in as many points G, and the line mn in as many points h; on the perpendicular LN set off all the distances Lp equal to the several distances Oh, cut on the radii by the directrix mn; then transfer the distances Op to the same radii produced to H, namely taking OH = Op; then shall the points H be so many points of the exterior curve, through all which points the bounding line being drawn with a steady hand, it will be as is seen in the figure to this example, which is accurately constructed and drawn by a scale to the dimensions above given, and which will extend infinitely along the directrix *mn*, this line being indeed an asymptote to the said curve.

The calculation in numbers is also equally easy and obvious. Thus, taking any given angle AOG, ob being $= \sqrt{OB^2 - OA^2}$, then $Lp = oh = ob \times \sec. AOG$, and hence $OH = Op = \sqrt{OL^2 + Lp^2} = \sqrt{OA^2 + Lp^2}$, which gives a point H in the curve. And the curve thus constructed gives the very same as the fig. p. 35, formed on the principles of prop. 6, as might be expected.

Examples of other curves, besides the circle, might be here taken, but the above case may suffice, as none of them are of a nature to be suitable for, or to hold good, in the construction of arches, at least for the ordinary purpose of bridges. Because, that in such arches, the parts do not endeavour to slide down in the oblique direction of the joints, both on account of the roughness or friction there, and because, when the parts are cemented together by the morter, or keyed together by pieces within side, the weights then all act perpendicular to the horizon, being each fixed to the other parts of the arch, after the manner supposed in the 9th and 10th propositions; and according to the examples to the latter of these, it will therefore be expedient to make such calculations as may occur in cases of real practice.

PROP. VIII.

When a curve is kept in equilibrio, in a vertical position, by loads or weights bearing on every point of it: then the load or vertical pressure on every point, is directly proportional to the product of the curvature at that point, and the square of the secant of the elevation above the horizon of the tangent to the curve at the same point, the radius being 1. *That is, the load or vertical pressure on any point* C, *is directly as the curvature at* C, *and as the square of the secant of the angle* bCH, *made by the tangent* bC *and the horizontal line* CH.



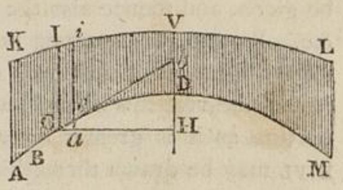
This property will be deduced as a corollary from the properties in the 2d and 3d propositions, according to the idea mentioned in the conclusion of the scholium there, by conceiving the bars or lines kept in equilibrio to become indefinitely small; for, by this means, those bars will form a continued curve line, after the manner of the arch stones in a bridge, constituting an arch of equilibration, by weights pressing vertically on every small or elementary part of the arch.

Now the consequence of the above idea, namely, of the bars becoming very small, and forming a continued curve, is, that the angle bCD becomes the angle of contact of the curve and tangent, and the angles bCH, DCH become equal to each other; consequently, the vertical load on the point C, which, in the 3d corol. prop. 3, was proportional to the sin. bCD × sec. bCH × sec. DCH, will be here proportional to the sin. bCD × sec^2. bCH, or as the angle bCD × sec^2. bCH, since a small angle (bCD) has the same proportion as its sine. But the angle of contact bCD, in any curve, is the measure of the curvature there; therefore, lastly, the vertical load or pressure, at any point C, in the curve of equilibration, is proportional to the curvature multiplied by the sec^2. of bCH; that is, proportional to the curvature at that point, and also to the square of the secant of the elevation of the curve or tangent above the horizon.

Corol.—Because the curvature at any point in a curve, is reciprocally proportional to the radius of curvature at that point; it follows, therefore, that the vertical load or weight on any point C, is as $\frac{\text{sec}^2.\ b\text{CH}}{r}$, where r denotes the radius of curvature at the point C; that is, directly proportional to the square of the secant of elevation, and inversely proportional to the radius of curvature to the same point.

PROP. IX.

When an upright wall, bounded by a curve beneath, is kept in equilibrio by the mutual weight and pressure of its parts and materials; then the height of the wall above every point of the curve, is directly proportional to the cube of the secant of elevation of the tangent to the curve there, and also directly proportional to the curvature at the same point, or else, which is the same thing, inversely proportional to the radius of curvature there.



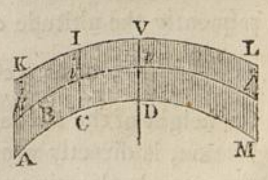
By the last proposition, the load or pressure on every elementary or small portion, cc, of the curve, is proportional to $\frac{\sec^2. b\text{CH}}{r}$. Now this load on every such small equal part of the arch, as cc, is a mass of solid matter cIic, incumbent on that part of the curve, and pressing it vertically; and which may be considered as made up of a number of equal heavy lines standing vertically on it; the number of which lines may be expressed by the breadth ca of the said pillar ci of heavy materials: but the breadth ca is $= \frac{\text{C}c}{\sec. \text{C}ca} = \frac{\text{C}c}{\sec. b\text{CH}}$, or as $\frac{1}{\sec. b\text{CH}}$, because the element cc is supposed given, or always of the same length, that is, ca is reciprocally as the secant of the angle of elevation. Hence then the vertical load, or ci, or $\frac{\text{CI}}{\sec. b\text{CH}}$, is as $\frac{\sec^2. b\text{CH}}{r}$; consequently the altitude CI of the wall AKLM, at the point C, is as $\frac{\sec^3. b\text{CH}}{r}$, or as $\sec^3. b\text{CH} \times$ curvature there. That is, the height of the wall above every part of the arch of equilibration, is directly proportional to the cube of the secant of the curve's elevation at that part, also directly proportional to the degree of curvature there, or else inversely as the radius of curvature at the same part.

Corollary 1.—Hence, if the form of the arch, or the nature of the inner curve ABCDM, be given; then the form or nature of the outer line KIL, bounding the top of the wall, or forming what is therefore called the extrados, may be found, so as that the intrados ABCDM shall be an arch of equilibration, or be in equilibrio in all its parts, by the weight or pressure of the superincumbent wall. For, since the arch or nature of the curve is given, by the supposition, the radius of curvature and position of the tangent, at every point of it, will be given, and thence also the proportions of the verticals CI, &c. So that, by assuming one of them, as the middle one VD for instance, or making it equal to an assigned length, the rest of the verticals will be found from it, and will be in proportion as it is greater or less; and then the extrados line KIVL may be drawn through all their extremities.

Or, on the other hand, if the extrados KIVL, or line bounding the top of the wall, be given; then the nature of the correspondent curve of equilibration ABCDM may be found out. And the manner of the practical derivation of both these curves, mutually the one from the other, will be shown in the following propositions.

Corollary 2.—If the intrados curve ABCD should be a circle; then the radius of curvature will be a constant quantity, and equal to the semidiameter of that circle; also the angle *b*CH will be always measured by the arc DC, from the vertex D of the curve; and then the height CI of the wall, will be every where proportional to the cube of the secant of the arch DC.

Corollary 3.—Hence also it follows, that if between the intrados and extrados curves, an intermediate curve *kivl*, be drawn through the middle of the wall, bisecting all the verticals DV, CI, &c, or indeed



dividing them in any ratio whatever, so as that it may be every where DV : Dv : : CI : Ci; then if ACDM be an arch of equilibration to the wall AKVLM, it will be an arch of equilibration to the inner wall AkvlM also.

PROP. X.

Having given the Intrados or Soffit, of a Balanced Arch; to find the Extrados. That is, having given the nature or form of an arch; from thence to find the nature of the line forming the top of the seperincumbent wall, by the pressure of which the arch is kept in equilibrio.

The solution of this problem is to be made out generally from the last proposition and its corollaries, by adopting general values of the lines there employed, which belong to all curves whatever: or otherwise by making use of the peculiar values proper to any individual curve, for the solution of particular cases.

For the general solution, in fig. pa. 41, KVL represents the extrados, the form of which is required, and ABCDM the given intrados or soffit of the arch, the vertex of which is D, and DV the height or thickness of the wall there, which is commonly a dimension that is known from the particular circumstances of the case. Now if we make the arch DC $= z$, its element Cc $= \dot{z}$, the absciss DH $= x$, its element $ca = \dot{x}$, the ordinate CH $= y$, its element C$a = \dot{y}$, the height or thickness of wall at the vertex DV $= a$, and the radius of curvature at any point C $= r$, that at the vertex D being $=$ R.

Then, because the height CI, at any point C, is as $\frac{\sec^3. b\text{CH or of } c\text{C}a}{r}$, by the last proposition, and because the secant of $c\text{C}a$ is $= \frac{\text{C}c}{\text{C}a} = \frac{\dot{z}}{\dot{y}}$, the radius being 1, therefore CI is as $\frac{\dot{z}^3}{r\dot{y}^3}$, or as $\frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{r\dot{y}^3}$, because $\dot{z} = \text{C}c = \sqrt{ca^2 + \text{C}a^2} = \sqrt{\dot{x}^2 + \dot{y}^2}$ or $(\dot{x}^2+\dot{y}^2)^{\frac{1}{2}}$.

Or, the general value of CI is $\frac{\dot{z}^3}{\dot{y}^3} \times \frac{Q}{r} = \frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{y}^3} \times \frac{Q}{r}$; where Q denotes a certain given or constant quantity, the value of which may be determined by making the general expression equal to a or DV, the height at the crown of the arch.

Corollary 1.—Because, at the vertex of the curve D, the angle of elevation is nothing, or its secant $\frac{Cc}{Ca} = \frac{\dot{z}}{\dot{y}} = 1$ the radius, and the radius of the curvature there being R; therefore the general expression for the height, becomes there DV $= a = \frac{Q}{R}$; consequently Q $= a$R, which is the general value of Q for all curves whatever, expressed in terms of the height a at the crown, and R the radius of curvature at the same point. Hence then, substituting this value of Q instead of it, the general expression or value of CI becomes $\frac{\dot{z}^3}{\dot{y}^3} \times \frac{aR}{r} = \frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{y}^3} \times \frac{aR}{r}$.

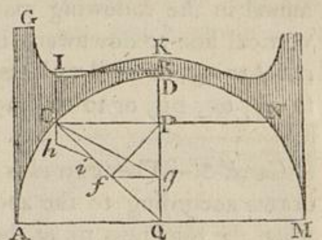
Corol. 2.—Because, in all curves that are referred to an axis, the general value of the radius of curvature r, is $= \frac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$; therefore, by substituting this value for r in the last expression, the general value of the height CI then becomes $\frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\dot{y}^3} \times a\text{R} = \frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\dot{y}^3} \times \text{Q}$, or $= \frac{-\dot{x}\ddot{y}}{\dot{y}^3} \times \text{Q}$ when $\dot{x}$ is constant.

For, as either x or y may be supposed to flow uniformly, and when, consequently, either of their second fluxions may be taken equal to nothing, which will cause one of the terms in the numerator of the above value of CI to vanish; therefore, by striking out either of those terms, and then exterminating either of the unknown quantities by means of the equation to the curve, the particular value of the height CI will be obtained: as is done in the following examples, except in some certain cases, where more peculiar methods may suit them better, as when the radius of curvature is a known quantity, &c.

EXAMPLE 1.

To find the Extrados of a Circular Arch.

That is, ACDM being a circular arc, of which AQ or QD is the semidiameter, Q the centre, and D the vertex of the given circular arch; also K the vertex of the extrados KIG, and the other lines as in the figure.



Making $a =$ DK, $r =$ AQ $=$ QD the radius of the circle, which is also equal to the radius of curvature throughout, or $r =$ R; also $x =$ DP, and $y =$ PC $=$ RI, and $z =$ the arch DC. Then, because $\frac{R}{r} = \frac{r}{r} = 1$, and $\frac{\dot{z}^3}{\dot{y}^3} =$ the cube of the secant of elevation at C, which is $= \frac{CQ^3 \text{ or } DQ^3}{PQ^3}$; therefore the general value of CI, in corol. 1, becomes CI $= \frac{DK \times DQ^3}{PQ^3} = (\frac{r}{r-x})^3 \times a = (\frac{r}{\sqrt{r^2 - y^2}})^3 a$; or, as $PQ^3 : DQ^3$ or $CQ^3 :: DK : CI$.

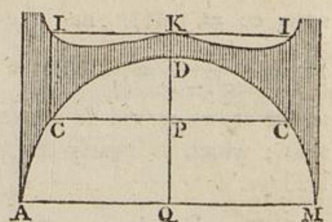
Corol. 1.—This expression for the value of CI, affords a very simple mode of calculation for the case of a circular arch; viz, to the constant logarithm of the height a, add triple the logarithm secant of the elevation, or of the arc DC; then the natural number answering to the sum will be the value of the height CI, at every point C.

Corol. 2.—It gives also a very simple construction by scale and compasses, which is as follows:—Join QC; draw P*f* perpendicular to QC, and *fg* perpendicular to QP; then shall Q*g* : QC :: QP^3 : QC^3; because, by similar triangles, Q*g* : Q*f* :: Q*f* : QP and :: QP : QC, or Q*g*, Q*f*, QP, QC are four terms in continued proportion, in which case the first Q*g* is to the fourth QC, as QP^3 to QC^3, the cube of the third to the cube of the fourth. Hence, if CI be taken a fourth proportional to Q*g*, QC, DK, it will be the length of the vertical line sought. And this fourth proportional will be easily determined in the following manner: viz, Join C*g*, and in the vertical line IC downward take C*h* = DK, and draw *hi* parallel to C*g*, so shall CI be equal to C*i* the fourth proportional to Q*g*, QC, DK, or to QP^3, QC^3, DK, as required.

Corol. 3.—The extrados line in this figure is accurately drawn according to the above construction and calculation, when the thickness DK at the crown is the exact 15th part of the span AM. It falls more and more below the horizontal line, from the crown all the way till the arch be between 30 and 40 degrees, where it takes a contrary flexure, tending upwards, passing the point I very obliquely, and thence rising very rapidly to an unlimited height, in an infinite curve, to which the vertical line AG is an asymptote; a circumstance which must always be the case with every curve, which, like AC, springs perpendicularly from the horizontal line AQM.

This curve cuts the horizontal line nearly over the point of 50 degrees. If DK were taken greater than the 15th part of AM, all the other vertical lines CI would be greater in the same proportion, and the curve KIG would cut the horizontal line drawn through K in some point still nearer to K; but the reverse, or farther off, if DK were taken less than the 15th part. Hence it appears, that a circular arch cannot be put in equilibrio by building on it up to a horizontal line, whatever its span may be, or whatever be the thickness at the crown. And consequently it may generally be inferred, that the circle is not a curve well suited to the purposes of a bridge which requires an outline quite horizontal, but may answer tolerably well when that line bends a little downwards, from the crown toward the extremities; and then a great variety of proportions between the thickness at the crown and the span of the arch might be assigned, which would put the circular arch in equilibrio, nearly.

Now these cases will happen in general when KR vanishes, or is of no length, and then CI must be equal to PK, or nearly so; with which general condition many particular cases may be found to agree nearly. But it may be proper here first to make out a general rule for such cases, which may be done in the following manner:



By the premises, the general value of CI being DK × $\sec^3$. DC, or as 1 : $\sec^3$ DC :: DK : CI; then, by taking CI = PK, in order to cause the outer curve KI to cross the horizontal line KI at the point I, that proportion becomes

1 : $\sec^3$ DC :: DK : PK or DK + DP,

or $\sec^3$ DC − 1 : 1 :: DP : DK $= \dfrac{\text{DP}}{\sec^3 \text{DC} - 1}$, the radius being 1.

Now, by taking the arch DC of various magnitudes, from DA or 90°, to O or nothing at D, the several thicknesses DK, at the crown, will be found by this theorem, corresponding to the several heights DP, or span CC, as here following, so as to make CDC a balanced arch very nearly. Thus,

1st. If DC be taken = DA or 90°: then its height is DQ = r, its span AM = $2r$, and its secant is infinite; consequently DK $= \dfrac{\text{DQ}}{\text{infin.}} = 0$. That is, the thickness at the crown comes out equal to nothing in this extreme case.

2d. If DC be taken = 75°: then its height DP = ·74118r, the span CC = 1·93185r, and the sec. DC = 3·8637; therefore DK $= \frac{\text{DP}}{\sec^3. - 1} = \cdot 01308r = \frac{\text{CC}}{148}$. That is, the thickness at the crown would be the 148th part of the span, being also much too small for common practice.

3d. If DC be taken $= 60°$: then its height DP $= \frac{1}{2}r$, the span CC $= r\sqrt{3}$, the sec. DC $= 2$; therefore DK $= \frac{\text{DP}}{2^3 - 1}$ $= \frac{1}{7}\text{DP} = \frac{1}{14}r = \frac{\text{CC}}{14\sqrt{3}} = \frac{\text{CC}}{24\cdot 2487} = \frac{\text{CC}}{24\frac{1}{4}}$ nearly. That is, the thickness at the crown would be rather less than the 24th part of the span: which is still too small in ordinary bridges.

4. If DC be taken $= 54°$: then its height DP $= \cdot 4122r$, the span CC $= 1\cdot 618r$, and the sec. DC $= 1\cdot 7013$; therefore DK $= \frac{\text{DP}}{\sec^3. - 1} = \cdot 10504r = \frac{\text{CC}}{15\cdot 41}$. That is, the thickness at the crown would be between the 15th and 16th part of the span; which is nearly the proportion allowed in common bridges.

5. If DC be taken $= 45°$: then its height DP $= r - \frac{1}{2}r\sqrt{2}$, the span CC $= r\sqrt{2}$, the sec. DC $= \sqrt{2}$; therefore DK $=$ $\frac{\text{DP}}{\sec^3. - 1} = \frac{1 - \frac{1}{2}\sqrt{2}}{2\sqrt{2} - 1}r = \frac{r}{2 + 3\sqrt{2}} = \frac{\text{CC}}{6 + 2\sqrt{2}} = \frac{\text{CC}}{8\cdot 8284} =$ $\frac{1}{9}$CC nearly. That is, the thickness at the crown would be more than the 9th part of the span: which in common cases is too much.

6. If DC be taken $= 30°$: then its height DP $= r - \frac{1}{2}r\sqrt{3}$, the span CC $= r$, the sec. DC $= \frac{2}{\sqrt{3}}$; therefore DK $=$

$$\frac{\text{DP}}{\sec^3. - 1} = \frac{r - \frac{1}{2}r\sqrt{3}}{\frac{8}{3\sqrt{3}} - 1} = \frac{6\sqrt{3} - 9}{16 - 6\sqrt{3}}r = \frac{6\sqrt{3} - 9}{16 - 6\sqrt{3}}\text{CC} = \frac{\text{CC}}{4\cdot 03} =$$

$\frac{1}{4}$CC nearly. That is, the thickness at the crown would then be almost the 4th part of the span.

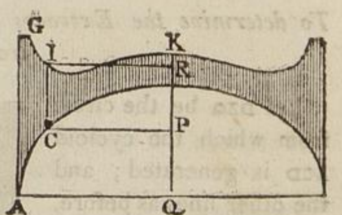
7. If DC be taken $= 15°$: then its height DP $= \cdot 03407r$,

the span CC $= \cdot 5176r$, and the sec. DC $= 1\cdot 0353$; therefore DK $= \frac{DP}{\sec^3. - 1} = \frac{DP}{\cdot 11} = 9DP = \cdot 3r = \frac{4}{7}$CC nearly, or $\frac{4}{7}$ of the span.

From all which it appears, that a whole arch CDC of about 108 or 110 degrees, is the part of the circle which may be used for most bridges with the least impropriety, the thickness at the crown being nearly the 16th part of the span, with a horizontal straight line at top.

EXAMPLE 2.

To determine the Extrados of an Elliptical Arch of Equilibration.



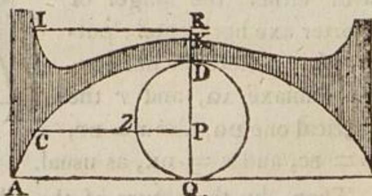
Suppose the curve in this figure to be a semiellipse, with either the longer or shorter axe horizontal: putting h to denote the horizontal semiaxe AQ, and r the vertical one DQ, also $x =$ DP, $y =$ PC, and $a =$ DK, as usual.

Then, by the nature of the ellipse, $r : h :: \sqrt{2rx - xx} : y$; therefore $y = \frac{h}{r}\sqrt{2rx - xx}$, and $\dot{y} = \frac{h\dot{x}}{r} \times \frac{r - x}{\sqrt{(2r - xx)}}$, also $\ddot{y} = \frac{-hr\dot{x}^2}{\sqrt{(2rx - xx)}}$ by making $\dot{x}$ constant. Hence the general value of CI, viz, $\frac{-\dot{x}\ddot{y}}{\dot{y}^3} \times$ Q, becomes $\frac{hr\dot{x}^3Q}{(2rx - xx)^{\frac{1}{2}}} \times \frac{r^3}{h^3\dot{x}^3} \times \frac{(2rx - xx)^{\frac{3}{2}}}{(r - x)^3} = \frac{r^4Q}{h^2(r - x)^3}$. But at the vertex of the curve D, where x is $= o$, this expression becomes only $\frac{rQ}{hh}$, which must be $=$ DK or a; therefore the value of Q is $= \frac{ahh}{r}$, which being substituted for it in the above general value of CI, this becomes $\text{CI} = \frac{ar^3}{(r-x)^3} = \frac{\text{DK} \times \text{DQ}^3}{\text{PQ}^3}$, which is the very same expression as the value of CI in the case of the circle in the former example, and which belongs equally to the ellipse in both positions, that is, both with the longer axe vertical, and with the shorter one vertical, as it is in the figure to this example.

Hence it appears, that the flat ellipse is more nearly balanced by a straight horizontal back or wall at top, than the circle is; but the circle more nearly than the sharp ellipse: the want of balance being least in the flat ellipse, but most in the sharp one, and in the circle a medium between the two.

EXAMPLE 3.

To determine the Extrados of a Cycloidal Arch of Equilibration.



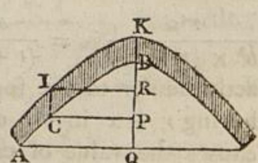
Let DZQ be the circle from which the cycloid ACD is generated; and the other lines as before.

Put $a = \text{DK}$, $x = \text{DP}$, and $y = \text{CP} = \text{IR}$, as usual; also put $r = \text{DQ}$ the diameter of the circle, and $z =$ the circular arc DZ. Then, by the nature of the cycloid, CZ is always equal to DZ $= z$; and, by the nature of the circle, PZ is $= \sqrt{rx - xx}$; therefore PC or y ($=$ CZ $+$ PZ) is $= z + \sqrt{rx - xx}$. Hence $\dot{y} = \dot{z} + \frac{\frac{1}{2}r - x}{\sqrt{(rx - xx)}} \times \dot{x}$; but $\dot{z}$ is $= \frac{\frac{1}{2}r\dot{x}}{\sqrt{(rx - xx)}}$ by the nature of the circle; therefore $\dot{y}$ is $= \frac{r - x}{\sqrt{(rx - xx)}} \times \dot{x} = \dot{x}\sqrt{\frac{r - x}{x}}$; then $\ddot{y} = \frac{-r\dot{x}^2}{2x\sqrt{(rx - xx)}}$, making $\dot{x}$ constant. Hence CI is $= \frac{-\dot{x}\ddot{y}\text{Q}}{\dot{y}^3} = \frac{\frac{1}{2}r\text{Q}}{(r - x)^2}$. But at the vertex D, $x = 0$, and

CI $= \frac{Q}{2r} = a$; therefore Q $= 2ar$; consequently the general value of CI is $(\frac{r}{r-x})^2 \times a = (\frac{DQ}{PQ})^2 \times$ DK; a formula which expresses the nature of the curve KI, for the extrados or back of a cycloidal curve of equilibration; a curve much resembling that for the circle and ellipse, in the two foregoing examples, as evidently appears by comparing the figures together, each of them being here accurately contracted. But this last figure, for the cycloid, seems to be rather better than either of those other two, as the extrados deviates rather less from a right line, and extends farther along before it bends upwards; and besides, the cycloidal arch is not deficient in either use or gracefulness.

EXAMPLE 4.

To determine the figure of the Extrados of a Parabolic Arch of Equilibration.

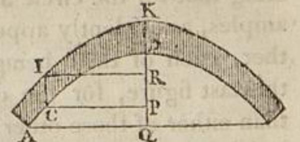


Putting, as before, $a =$ KD, $r =$ DQ, $h =$ AQ, $x =$ DP, and $y =$ PC $=$ RI. Then, by the nature of the curve, $hh : yy :: r : x = \frac{ryy}{hh}$; hence $\dot{x} = \frac{2ry\dot{y}}{hh}$, and $\ddot{x} = \frac{2r\dot{y}^2}{hh}$, by making $\dot{y}$ constant. Then CI $= \frac{\ddot{x}}{\dot{y}^2} \times$ Q is $= \frac{2r\text{Q}}{hh} =$ a constant quantity $= a$; that is, CI is every where equal to KD.

Consequently KR is $=$ DP; and since RI is $=$ PC, it is evident that KI is the same parabolic curve with DC, and may be placed any height above it, always producing an arch of equilibration.

EXAMPLE 5.

To find the figure of the Extrados for an Hyperbolic Arch of Equilibration.



Here putting, as before, a = KD, r = the semi-transverse, and h = the horizontal or semi-conjugate axe, also x = DP, and y = PC = RI. Then, by the nature of the hyperbola, $y = \frac{h}{r}\sqrt{2rx + xx}$; hence $\dot{y} = \frac{h\dot{x}}{r} \times \frac{r + x}{\sqrt{(2rx + xx)}}$, and, by making $\dot{x}$ constant, $\ddot{y} = \frac{-hr\dot{x}^2}{(2rx+xx)^{\frac{3}{2}}}$.

Therefore CI or $\frac{-\dot{x}\ddot{y}}{\dot{y}^3} \times$ Q is $= \frac{r^4 Q}{h^2 \times (r + x)^3}$. But in the vertex D, where $x = 0$, this expression becomes

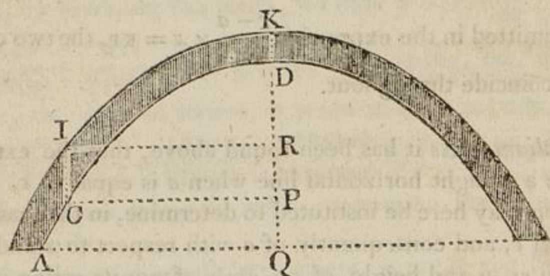
$\frac{rQ}{hh} = a$; hence $Q = \frac{ahh}{r}$, and consequently CI or

$\frac{r^4 Q}{h^2 \times (r + x)^3}$ is $= \frac{ar^3}{(r+x)^3} = (\frac{r}{r + x})^3 \times a$, which is exactly similar to the formula for the circle and ellipse, only having $r + x$ in the denominator, instead of $r - x$, which causes the value of CI to become always less and less, as the point C is taken farther from the vertex D.

In this hyperbolic arch then, it is evident that the extrados KI continually approaches nearer and nearer to the intrados; whereas in the circular and elliptic arches, it goes off continually farther and farther from it; while in the parabola, the two curves keep always at the same distance. Observing, however, that, by the distance between the two curves, in all these cases, is meant their distance in the vertical direction.

EXAMPLE 6.

To find the Extrados for a Catenarian Arch of Equilibration.



Let $a =$ KD, $x =$ DP, and $y =$ PC $=$ RI, as before; also let c denote the constant tension of the curve at the vertex. Then, by the nature of the catenary, y is $= c \times$ hyp. log. of $\frac{c + x + \sqrt{2cx + xx}}{c}$; hence, taking the fluxions, we have $\dot{y} = \frac{c\dot{x}}{\sqrt{(2cx + xx)}}$, and $\ddot{y} = -c\dot{x}^2 \times \frac{c + x}{(2cx + xx)^{\frac{3}{2}}}$, by making $\dot{x}$ constant. Therefore CI, or $\frac{-\dot{x}\ddot{y}}{\dot{y}^3} \times$ Q, is $= \frac{c+x}{cc} \times$ Q. But at the vertex x is $= 0$, and CI $= a = \frac{Q}{c}$; consequently Q is $= ac$. This being written for it, there results CI $=$ $\frac{c+x}{c} \times a = a + \frac{ax}{c}$. And the same formula comes out for the logarithmic curve. Hence, for the nature of the curve KI, we have KR $= (a + x -$ CI $=) x - \frac{ax}{c} = \frac{c-a}{c} \times x$.

Corol.—And hence the abscissa DP, of the inner or soffit curve, is to the abscissa KR, of the exterior one, always in the constant proportion of c to $c - a$. So that, when a is less than c, R and the curve KI lie below the horizontal line; but when a is greater than c, they lie above it; and when a is equal to c, KR is always equal to nothing, and KI, or the extrados, coincides with the horizontal line. As a diminishes, the line KI approaches always nearer to DC in all its parts, till, when a entirely vanishes, or is so small in respect of c as to be omitted in the expression $\frac{c-a}{c} \times x = \text{KR}$, the two curves quite coincide throughout.

Scholium.—As it has been found above, that the extrados will be a straight horizontal line when a is equal to c, a calculation may here be instituted to determine, in that case, the value of c, and consequently of a with respect to x and y, or a given span and height of an arch of equilibration in that case. Now the equation to the curve expressed in terms of c, x, and y, is $y = c \times \text{hyp. log. of } \frac{c + x + \sqrt{2cx + xx}}{c}$; and when x and y are given, the value of c may be found from this equation, by the method of trial and error. But as the process would be at best but a tedious one, and perhaps the method not easy in this case to be practised by every person, we may here investigate a series for finding the value of c from those of x and y in a direct manner. Since then $y = c \times \text{hyp. log. of } \frac{c + x + \sqrt{2cx + xx}}{c}$, by taking the fluxion of this equation, we have

$\dot{y} = \frac{c\dot{x}}{\sqrt{(2cx + xx)}} = \frac{\frac{1}{2}d\dot{x}}{\sqrt{(dx + xx)}}$, by writing d for $2c$; and by expanding this expression into a series, it becomes

$\dot{y} = \frac{1}{2}\dot{x}\sqrt{\frac{d}{x}} \times (1 - \frac{x}{2d} + \frac{1.3x^2}{2.4d^2} - \frac{1.3.5x^3}{2.4.6d^3}$ &c); and, by taking the fluents, we have $y = \sqrt{dx} \times (1 - \frac{x}{2.3d} + \frac{1.3x^2}{2.4.5d^2} - \frac{1.3.5x^3}{2.4.6.7d^3} + \frac{1.3.5.7x^4}{2.4.6.8.9d^4}$ &c); hence, dividing by x, we have $\frac{y}{x} = \sqrt{\frac{d}{x}} \times (1 - \frac{x}{2.3d} + \frac{1.3x^2}{2.4.5d^2} - \frac{1.3.5x^3}{2.4.6.7d^3} +$

$\frac{1.3.5.7x^4}{2.4.6.8.9d^4}$ &c); or, by writing v for $\frac{y}{x}$, and w for $\sqrt{\frac{d}{x}}$, it is $v = w - \frac{1}{2.3w} + \frac{1.3}{2.4.5w^3} - \frac{1.3.5}{2.4.6.7w^5} + \frac{1.3.5.7}{2.4.6.8.9w^7}$ &c.

Then, by reverting this series, we have $w = v + \frac{1}{6v} - \frac{37}{360v^3} + \frac{547}{5040v^5} - \frac{337}{5600v^7}$ &c. Hence, by squaring, &c, and restoring the original letters, it is $(\frac{1}{2}d = \frac{1}{2}xw^2 =)\ c = \frac{1}{2}x \times (\frac{y^2}{x^2} + \frac{1}{3} - \frac{8x^2}{45y^2} + \frac{691x^4}{3780y^4} - \frac{23851x^6}{453600y^6}$ &c), where a few of the first terms are sufficient to determine the value of c pretty nearly.

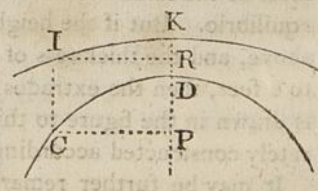
Now, for an example in numbers, suppose the height of the arch to be 40 feet, and its span 100, which are nearly the dimensions of the middle arch of Blackfriars Bridge at London. Then $x = 40$, and $y = 50$; which being substituted for them in this series, it gives $c = 36{\cdot}88$ feet nearly. So that, to have made that arch a catenarian one, with a straight line above, the top of the arch must have been almost of the immense thickness of 37 feet, to have kept it in equilibrio. But if the height and span be 40 and 100 feet, as above, and the thickness of the arch at top be assumed equal to 6 feet, then the extrados will not be a right line, but as it is drawn in the figure to this example, which figure is accurately constructed according to these dimensions.

It may be further remarked, that the curves in these last three examples, viz, the parabola, hyperbola, and catenary, are all very improper for the arches of a bridge consisting of several arches; because it is evident from their figures, which are all constructed from a scale, that all the building or filling up of the flanks of the arches will tend to destroy the equilibrium of them. But in a bridge of one single arch, whose extrados or back rises pretty much from the spring to the top, one of these figures will answer better than any of the former ones.—Other examples of known curves might be given; but those that have been here noticed, seem to be the fittest for real practice; and there is a sufficient variety among them, to suit the various circumstances of convenience, strength, and beauty, that may be desired.

We may now proceed to another general problem, which is the reverse of the last, and is, to determine the figure of the intrados for any given figure of the extrados, so that the arch may be in equilibrio in all its parts. This is a more difficult problem than the former, and the more useful one also. Here commonly, that the roadway may be of easy and regular ascent, we are confined to an outline nearly horizontal, to which the curve of the soffit or inner arch must be adapted.

PROP. XI.

Having the Extrados given; to find the Intrados. That is, having given the nature or form of a line, bounding the top of a wall above an arch; to determine the figure of the arch, so that, by the pressure of the superincumbent wall, the whole may remain in equilibrio.



Putting a = DK the thickness of the arch at top, x = DP the absciss of the required intrados arch DC, u = KR the corresponding absciss of the given extrados KI, and y = PC = RI their equal ordinates.

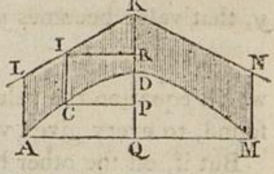
Then, by the last prop. CI is $= \frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\dot{y}^3} \times Q$; but CI is also evidently equal to $a + x - u$; therefore $a + x - u$ is $= \frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\dot{y}^3} \times Q = \frac{Q}{\dot{y}} \times$ the fluxion of $\frac{\dot{x}}{\dot{y}}$; where Q is a constant quantity, as used in the last proposition, and is always to be determined from the nature or conditions of each particular case, commonly indeed by taking the real value of CI, viz, DK or a at the vertex of the curve.

Hence then, by substituting, in this equation, the given value of u instead of it, as expressed in terms of y, the resulting equation will then involve only x and y, together with their first and second fluxions, besides constant quantities. And from it the relation between x and y themselves may be found, by the application of such methods as may seem to be best adapted to the particular form of the given equation to the extrados. In general, a proper series for the value of x in terms of y is to be assumed with indeterminate coefficients; which series being put into fluxions, striking out of every term the fluxion of y; and the result put into fluxions again, striking out from every term of this also the fluxion of y; the last expression drawn into Q being equated to $a + x - u$, there will be produced an equation, from which may be found the values of the coefficients of the terms in the assumed value of x.

Fortunately however, the process is more simple and easy in the most common and useful cases, than might at first be expected from this general method, viz, when the extrados is a straight line, even when it is oblique, and still more when it is horizontal; two cases to which we shall now proceed to apply the general method, in the following examples.

EXAMPLE 1.

To find an Arch of Equilibration when the Extrados is a straight line, oblique or inclined.



In this case, the extrados will have a resemblance to the sloping roof of a house, as in the annexed figure, and is often used in the case of gunpowder magazines. Here employing the notation as in the proposition, the general equation there is CI, or w, $= a + x - u = Q \times \frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\dot{y}^3}$, or $= Q \times \frac{\ddot{x}}{\dot{y}^2}$, supposing $\dot{x}$

a constant quantity. But KR or u is $= ty$, if t be put to denote the tangent of the given angle of elevation KIR, to radius 1; and then the equation is $w = a + x - ty = \frac{Q\ddot{x}}{\dot{y}^2}$.

But the fluxion of the equation $w = a + x - ty$, is $\dot{w} = \dot{x} - t\dot{y}$, and the second fluxion is $\ddot{w} = \ddot{x}$; therefore the general equation becomes $w = \frac{Q\ddot{w}}{\dot{y}^2}$; and hence $w\dot{w} = \frac{Q\dot{w}\ddot{w}}{\dot{y}^2}$, the fluent of which gives $w^2 = \frac{Q\dot{w}^2}{\dot{y}^2}$: but at D the value of w is $= a$, and $\dot{w} = 0$, because the curve at D is parallel to KI; therefore the correct fluent is $w^2 - a^2 = \frac{Q\dot{w}^2}{\dot{y}^2}$. Hence then $\dot{y}^2 = \frac{Q\dot{w}^2}{w^2 - a^2}$, or $\dot{y} = \frac{\dot{w}\sqrt{Q}}{\sqrt{w^2 - a^2}}$; the correct fluent of which gives $y = \sqrt{Q} \times$ hyp. log. of $\frac{w + \sqrt{w^2 - a^2}}{a}$.

Now, when the vertical line CI is at the position AL, then $w =$ CI becomes AL $=$ the given quantity c suppose, and $y =$ AQ $= h$, in which case the last equation becomes $h = \sqrt{Q} \times$ hyp. log. of $\frac{c + \sqrt{c^2 - a^2}}{a}$; hence it is found, that the value of the constant quantity $\sqrt{Q}$ is $\frac{h}{\text{h. l. of} \frac{c + \sqrt{c^2 - a^2}}{a}}$; which being substituted for it in the above general value of y, that value becomes $y = h \times \frac{\text{log. of} \frac{w + \sqrt{w^2 - a^2}}{a}}{\text{log. of} \frac{c + \sqrt{c^2 - a^2}}{a}}$; from which equation the value of the ordinate CP may always be found, to every given value of the vertical CI.

But if, on the other hand, PC be given, to find CI, which will be the more convenient way, it may be found in the following manner: Put A = the log. of a, and $e = \frac{1}{h} \times$ log. of

$\frac{c + \sqrt{c^2 - a^2}}{a}$; then the above equation gives $\text{C}y + \text{A} =$ co log. of $(w + \sqrt{w^2 - a^2})$; again, put $n =$ the number whose log. is $\text{C}y + \text{A}$; then $n = w + \sqrt{w^2 - a^2}$; and hence $w = \frac{a^2 + n^2}{2n} = \text{CI}$.

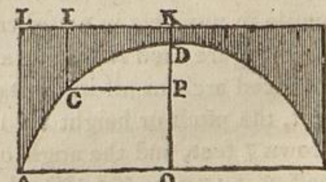
This example is more peculiarly adapted to the use of magazines for gunpowder, which are usually made in the manner represented in the figure above, that is in regard to their roof, for the inner curve itself has commonly been made a semicircle. But it is a constant observation, that after the centering of semicircular arches is struck, they settle at the crown, and rise up at the flanks, even with a straight horizontal extrados, and still much more so in powder magazines, where the outside at top is formed, like the roof of a house, by two inclined planes joining in an angle, or ridge, over the top of the arch, to give a proper descent to the rain; which effects are exactly what might be expected from a contemplation of the true theory of arches. Now this shrinking of the arches must be attended with very bad consequences, by breaking the texture of the cement, after it has in some degree been dried, and also by opening the joints of the vousoirs at one end; consequently the application of the formula above investigated must be accompanied with beneficial effects. It may be useful therefore to give here an example in numbers in a real case of that nature. If the foregoing figure then represent a transverse vertical section of a balanced arch in all its parts, in which the span AM is 20 feet, the pitch or height DQ 10 feet, the thickness DK at the crown 7 feet, and the angle of the ridge LKN 112° 37′, or the half of it LKD = 56° 18′$\frac{1}{2}$, the complement of which, or the elevation KIR, is 33° 41′$\frac{1}{2}$, the tangent of which is $= \frac{2}{3}$, which will therefore be the value of t in the investigation above. The values of the other letters will be as follows, viz, $\text{DK} = a = 7$; $\text{AQ} = h = 10$; $\text{DQ} = r = 10$; $\text{AL} = c = \frac{31}{3} = 10\frac{1}{3}$;

A $= \log.$ of $7 = \cdot 8450980$; c $= \frac{1}{h} \times \log.$ of $\frac{c + \sqrt{c^2 - a^2}}{a} = \frac{1}{10}$ log. of $\frac{31 + \sqrt{520}}{21} = \frac{1}{10}$ log. of $2 \cdot 56207 = \cdot 0408591$; c$y$ + A $= \cdot 0408591y + \cdot 8450980 =$ the log. of n. From the general equation then, viz, CI $= w = \frac{a^2 + n^2}{2n}$, by assuming y successively equal to 1, 2, 3, 4, &c, and thence finding the corresponding values of cy + A or ·0408591 + ·8450980, and to these, as common logs, taking out the corresponding natural numbers, or values of n; then the above theorem will give the several values of w or CI, as they are here arranged in the annexed table, from which the figure of the curve is to be constructed, by finding so many points in it.

Val. of y or CP.	Val. of w or CI
1	7·0310
2	7·1243
3	7·2806
4	7·5015
5	7·7838
6	8·1452
7	8·5737
8	9·0781
9	9·6628
10	10·3333

EXAMPLE 2.

To find an Arch of Equilibration whose Extrados shall be a Horizontal line.



The process for this case differs in nothing from that in the former example, but in substituting the horizontal line of extrados KI, instead of the oblique one, by which the angle DKI becomes a right angle, therefore the angle KIR, in the former example, vanishes, and consequently its tangent also, that is, the value of t, in the last example, becomes nothing in this: all the other letters and the formula being the very same.

For an example therefore in numbers, let us suppose the span of the arch to be 100 feet, the pitch or height 40 feet, and thickness at the crown 6 feet, which are nearly the dimensions of the centre arch in Blackfriars bridge: then the values of the several letters will be as follows, viz, AQ = $h = 50$; DQ $= r = 40$; DK $= a = 6$; AL $= c = 46$. Hence the hyp. log. of $\frac{c + \sqrt{c^2 - a^2}}{a}$ = hyp. log. of $\frac{46 + 4\sqrt{130}}{6}$ = hyp. log. of 15·26784 = 2·7257487; by which dividing h or 50, the quotient is 18·343584. So that the ordinate y will be constantly, in that case, equal to 18·343584 × hyp. log. of $\frac{w + \sqrt{w^2 - a^2}}{a}$. Also $\frac{1}{18 \cdot 343584}$ = ·05451497 is = c, and A = hyp. log. of 6 = 1·7917594; therefore n is = the number whose hyp. log. is cy + A or ·05451497 y + 1·7917594. Hence, by assuming several values of the letter y, which is = CP or IK, the corresponding values of n will be found as above, and then those of w or CI from the final general equation $w = \frac{a^2 + n^2}{2a} = \frac{36 + n^2}{12} = 3 + \frac{1}{12} n^2$. And in this manner were calculated the numbers in the following table; from which the curve being constructed, it will be as appears in the figure to the example.

And thus we have an arch in equilibrium in all its parts, and its top a straight line, as is generally required in most bridges; or at least they are so near a horizontal line, that their difference from it will cause little or no sensible ill consequence. It is also both of a graceful figure, and of a convenient form for the passage through it. So that no reasonable objection can be offered against its adoption in works of consequence, on account of its mechanical excellency.

The Table for Constructing the Curve in this Example.

Value of KI	Value of IC	Value of KI	Value of IC	Value of KI	Value of IC	Value of KI	Value of IC	Value of KI	Value of IC
0	6·000	15	8·120	24	11·911	33	18·627	42	29·919
2	6·035	16	8·430	25	12·489	34	19·617	43	31·563
4	6 144	17	8·766	26	13·106	35	20·665	44	33·299
6	6·324	18	9·168	27	13·761	36	21·774	45	35·135
8	6·580	19	9·517	28	14·457	37	22·948	46	37 075
10	6·914	20	9·934	29	15·196	38	24·190	47	39·126
12	7·330	21	10·381	30	15·980	39	25·505	48	41·293
13	7·571	22	10·858	31	16·811	40	26·894	49	43·581
14	7·834	23	11·368	32	17·693	41	28·364	50	46·000

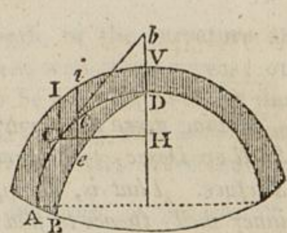
The above numbers may either be feet, or any other lengths, of which DQ is 40 and QA is 50. But when DQ is to QA in any other proportion than that of 4 to 5, or when DK is not to DQ as 6 to 40 or 3 to 20; then the above numbers will not answer; but others must be found by the same rule, to construct the curve by. In the beginning of the table, as far as 12, the value of KI is made to differ by 2, because the value of CI in that part increases so very slowly. Afterwards they differ by units or 1.

Other examples of given extrados might be taken; but as there can scarcely ever be any real occasion for them, and as the trouble of calculation would be, in most cases, very great, they are omitted.

As the theory for arch vaults, before laid down, will so easily apply to the arches for domes or cupolas also, a proposition or two may be here added for that purpose, as follows.

PROP. XII.

When a regular Concave Surface Dome, or Vault, formed by the rotation of a curve turned about its axis, is kept in equilibrio by the pressure of a solid wall built on every part of it; then the Height of the wall over any part, is directly proportional to the cube of the secant of elevation there, and inversely proportional to the radius of curvature, and to the diameter or width of the dome at the same part.



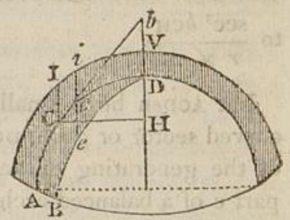
That is, VI being the form of the exterior surface of a balanced shell, the interior surface of which is formed by the rotation of the curve DCA about its axis DH; the elevation of any part C being the angle *b*CH, and CH the ordinate or semidiameter of the dome at the point C, also *r* the radius of curvature to the same point: then the height or vertical thickness of the shell over the point C, or CI, is proportional to $\frac{\sec^3 b\text{CH}}{r \times \text{CH}}$.

Let ACD*e*B be a small part of the inner surface, like a curved sector or gore, DCA and D*e*B being two near positions of the generating curve. Now the vertical load on any part C of a balanced arch, in a shell or dome, in the present case, is a solid pillar, C*i*, whose height is CI, its breadth *ca*, and thickness *ce*, and consequently is $= \text{CI} \times ca \times ce$. But *ca* is as $\frac{\text{CH}}{cb}$ or as $\frac{1}{\sec. b\text{CH}}$; and *ce* is always in the same proportion as CH; therefore the pillar C*i*, or $\text{CI} \times ca \times ce$ is as $\frac{\text{CI} \times \text{CH}}{\sec. b\text{CH}}$; which load, by the 8th prop. is also proportional to $\frac{\sec^2. b\text{CH}}{r}$; therefore $\frac{\text{CI} \times \text{CH}}{\sec. b\text{CH}}$ is as $\frac{\sec^2. b\text{CH}}{r}$; consequently the height CI is as $\frac{\sec^3. b\text{CH}}{r \times \text{CH}}$. That is, the vertical height of the wall over every part of a balanced shell, or dome, or vault, is directly as the cube of the secant of the curve's elevation at that part, and inversely as the radius of curvature, and also inversely as the width of the dome at the same place.

And here may be also understood several corollaries and observations exactly similar to those to the 3d and the 9th propositions, and which therefore need not be repeated in this place.

PROP. XIII.

Having given the form of the Inner Surface of a balanced Shell or Dome; to determine that of the Exterior or Outer Surface. That is, having given the nature or form of an inner shell; thence to find the nature of the outer or bounding surface of the superincumbent wall, by the pressure of which the shell is kept in equilibrio.



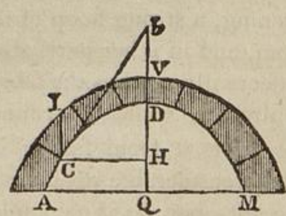
By reasoning here exactly as in the 10th proposition, it will be found that the general value of the height CI of the wall, will be proportional to the following forms or quantities, viz,

CI is either as $\frac{\sec^3 .\ b\text{CH}}{r \times \text{CH}}$, or as $\frac{\dot{z}^3}{ry\dot{y}^3}$, or as $\frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{ry\dot{y}^3}$, or as $\frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{ry\dot{y}^3}$, or as $\frac{-\dot{x}\ddot{y}}{y\dot{y}^3}$ when $\dot{x}$ is considered as invariable, or as $\frac{\ddot{x}}{y\dot{y}^2}$ when $\dot{y}$ is invariable: in which the letters have the usual values, namely, $x = $ DH the absciss, $y = $ CH the ordinate, and $z = $ DC the curve, also r the radius of curvature at the point C. Or the general value of CI will be equal to any of these forms multiplied by a certain constant quantity Q, the particular value of which is always to be determined by putting the general value of CI equal to the given thickness of the shell, either at the crown, or at some other particular place, where that value may happen to be known or given.

Corol.—From this, and the foregoing prop. we may infer this general observation, namely, that no curve can produce the figure of a true or exact balanced dome or cupola, unless that curve be of such a nature as to have its radius of curvature at the vertex of an infinite length, or the curvature at the vertex nothing; which is the case with some curves; or unless the thickness at the crown be infinite. For, at the vertex, the angle of elevation *b*CH is nothing, and the secant = 1; the ordinate CH is there nothing also; therefore the general expression, $\text{CI} = \frac{\sec^3 . b\text{CH}}{r \times \text{CH}}$, becomes, at the vertex, $\text{DV} = \frac{1}{r \times 0} = \frac{1}{0} =$ infinite, that is DV must be infinite, if r be a finite quantity.

Or, if DV be finite, as suppose = A; then $a = \frac{1}{r \times 0}$, or $r = \frac{1}{a \times 0} = \frac{1}{0} =$ infinite, when a is finite. That is, the radius of curvature at the vertex must be infinite when the height there is finite or given; or, on the other hand, the height or pressure at the vertex must be infinite, when the radius of curvature there is a finite or given quantity, to have the shell truly balanced. Of this nature there are several curves, of the parabolic kind in particular, of a form both convenient and graceful, such as the cubical parabola in the following example.

EXAMPLE.



Taking, for an example, the curve DC of the cubical parabola, so called because its abscisses are proportional to the cubes of their ordinates. Thus, putting x = DH the absciss, y = CH the ordinate, and a the parameter, or a given quantity; then the equation to the curve is $ax = y^3$. Hence, taking the fluxions, we obtain $\dot{x} = \frac{3y^2\dot{y}}{a}$, and $\ddot{x} = \frac{6y\dot{y}^2}{a}$, when $\dot{y}$ is considered as invariable This value of $\ddot{x}$ being substituted for it in the general value of the height CI, viz, $\frac{\ddot{x}}{y\dot{y}^2}$, this becomes $CI = \frac{6y\dot{y}^2}{ay\dot{y}^2} = \frac{6}{a}$; that is, any given or constant quantity. Consequently the outer curve is the same as the inner, but placed in a higher position, as they appear in the figure to this example, where the curves are accurately constructed to a particular scale, when the greatest width AM is 80 feet, and the height DQ is 64 feet.

The foregoing principle for balancing dome vaults, it must be understood, is quite independent of the aid it receives from the circular or other form of its contour, in which indeed consists its great strength and stability. For, from this shape it happens, that the inside or outer one, in the vertical section, may take any form whatever, either convex outwards, as is usual in rotund domes, or a straight side, as in the cone of tile kilns or the pyramidal spire, or even concave outwards and convex inwards. For, by making all the coursing joints of masonry, quite around, not flat or horizontal, but everywhere perpendicular to the face, and all the vertical joints tending or pointing to the axis, all the stones or bricks, &c, will act as wedges in a round curb, and cannot possibly come down, or fall inwards, unless the component parts could be crushed to powder, or the bottom circular course burst outwards. To prevent this from happening, a strong hoop of iron may be passed round the bottom, and in other parts also, in works of consequence, which effectually secures the fabric from bursting open, or flying outwards, while the round form, like a curb, as securely prevents it from falling inwards. Hence too it happens, that considerable openings may be cut in the sides, or it may be left open, as if incomplete, at top, and over the opening may be erected any other figure, whether lantern or spire, &c, either for use or ornament.

GENERAL SCHOLIUM.

In the foregoing propositions have been delivered the chief variety of ways for constructing the arches of bridges, so as they may be in equilibrio or balanced in themselves. There are three of these different methods; first, that which is derived from the consideration of the equilibrium produced by the mutual thrusts, weights and pressures of the arch stones, supposing them prevented from sliding on each other at the oblique joints, either by their roughness and friction, or by the cement, or stone locks, or iron bars let into every adjacent pair of stones; which give the arch the effect of one compacted frame, pressed on vertically by the weight of the superincumbent load of wall above it: which seems to be the true and genuine way of considering the action of that load on the arch.

The second method, is that in which the balanced arch is computed on the supposition that the arch stones have their butting sides perfectly smooth, and at free liberty to slide on each other. A method which is but little insisted on, as it is founded on a supposition which is neither in nature nor art, and which can never take place in any real construction of an arch.

The third method, is that which has for its principle the catenarian or festoon arch, formed by the suspension of a slack chain or cord, by its two ends, and afterwards inverted. This idea it seems was first proposed by Dr. Hooke, near the latter part of the 17th century, when the Newtonian mathematics prepared the way to true mechanical science. This is a strictly just and useful principle, and may be most easily extended to every case that can happen in practice. At first indeed the idea had nothing more in view than the balancing of the single or thin arch, formed by the voussoirs only, as the catenarian curve, formed by a simple chain or cord, can aim at nothing further than the balancing of that simple string of arch stones, without any other wall to fill up the flanks, &c. This principle was also neatly treated of by Delahire, in prop. 123, 124, 125 of his Traité de Mecanique, published in 1695. But the same principle has been lately acted on, and extended much further, by professor Robison of Edinburgh, namely, by making thus a festoon arch balancing, not only the simple string of voussoirs, but also the whole load of the superincumbent wall, of any proposed form whatever. This method, so easy in its practical operation, depends on, and is easily deduced from the first, or that which balances the arch by the mutual thrusts and pressures of the parts; by showing that these forces, of mutual pressure of the parts, are exactly equal and opposite to those by which they pull or draw each other in the case of suspension.

It is true that the equilibrium which any theory establishes, is of so delicate a nature, by supposing the parts to touch only in single points, that it may be called a tottering equilibrium, since any other weight or force added at any part would press the arch out of its true balanced form, and, by shifting the points of contact of the parts, bring the whole down to the ground, if it were not that the arch stones have some considerable length, by which a stability is ensured, as the altered figure will find new points of contact, where the action of the parts will principally bear, and through all which points a new curve line may be conceived to pass, as the catenary or festoon balanced arch. And hence it follows, that the longer the butting joints or arch stones are, the more stable and secure the whole fabric will be; since this circumstance will allow of the more change either in the figure of the arch, or the true catenarian points of bearing or thrust, and yet have a competent substance of solid stone to sustain the great force of such actions. It is therefore of the greatest importance to have the arch stones made as long as may be, consistent with economy, and the other circumstances of the fabric. And this was the great use of the ribs that were employed in the old English architecture, the great projections of which augmented considerably the stiffness of the whole, and enabled the architects to make use of comparatively very small stones in the other parts of the work. This contrivance we find has been used in constructing roofs, as well as in bridges; the few old remaining ones of these we see have been constructed and strengthened by these ribs of long and large stones. It would therefore be perhaps the safest and firmest way, to give the whole masonry of the wall, over the arch stones, the same position of joints as these stones themselves have, namely, not in horizontal courses, but everywhere the joints in the direction perpendicular to the curve of the arch, quite up to the top or road way; as we see indeed has been practised in the face of the masonry at Westminster bridge. For, by this means, the whole has the effect of arch stones, considered as extended the whole length, from the soffit of the arch, all the distance up to the road way: thus ensuring a strength and safety so complete, as to render even considerable deviations from the theory of a balanced arch of no material bad effect whatever.

SECTION III.

OF THE PIERS.

When an arch is supposed to stand alone, and well balanced, it is necessary that its piers or abutments should be at least sufficiently firm and massive to resist completely the shoot, drift, or horizontal push of the arch. For should the pier yield in the least to this drift, and be pushed aside, the arch must infallibly fall down. It is therefore essential that every arch should have its abutments properly adapted to resist effectually its shoot. And the same precaution ought also to be employed in a string or series of arches, such as an arcade, or a long bridge composed of several openings: for though, in these cases, the arches may be supposed to sustain mutually each other's thrust, while they are all standing, and to require only a slender pier between every adjacent pair of arches, to serve as a thin plane between their mutual pushes, like the ridge board between the butting ends of the rafters in the roof of a house; yet provision should be made against any possible accident that may happen to any one of the arches in the string, so as that any of them may be supposed cut open, or to fall down, and yet not affect the adjacent ones, but leave them standing firm and independent, sustained by their own piers alone. For otherwise, should the arches be made in a string as it were, all dependent on each other for support, then on an accident befalling any one arch, the entire series of arches must follow it, and the whole fabric come down.

Prudent architects therefore take care to employ various means of constructing their piers to be, as they expect, sufficiently stable and firm, to sustain the shoot of the arches; without however being always certain of the just and adequate effect. For this reason it sometimes happens, that their piers are made too slender for perfect safety, and sometimes indeed, erring on the other hand, they are made unnecessarily thick and massive; a mistake which, to say nothing of the ungraceful appearance, both enhances the expence, and also impedes the free and easy passage of the water and navigation, by occupying too much of the breadth of the river, by such loads of solid masonry. It is therefore intended, in this section, to give rules and examples for computing nearly the proper thickness and weight of a pier, so as to be an exact balance to the shoot of the arch; that by then giving it a very little more thickness in practice, a security is provided against any accidental and extraneous effort.

But this equilibrium is not easily or certainly to be effected: it is by all authors attempted, though not always justly, by determining the thickness of the piers such, that the resistance of its weight to being overset, may be at least equal to the force of the shoot or drift of the arch against it. This principle is obvious enough; but then all authors have not agreed in the method of estimating the value of this last force in particular. Some have determined this point on supposition that the wedges or arch stones are perfectly smooth and unconnected with each other; while others have supposed them so firmly connected, as to form the arch into a solid mass, acting like one rigid body only. It is true, and it has been proved in the beginning of this work, that in an arch of equilibration, formed of parts properly disposed, whether of wedges, or of vertical pieces, the horizontal push or shoot is constantly the same quantity in every part of the arch; being to the weight of the arch above that part, as radius to the tangent of the elevation of that part of the arch above the horizontal line: from which circumstance some persons have imagined that, by computing the shoot or drift for any small given part, as at the key stone for instance, which can easily be done, that will be a sufficient measure or value of the whole; then by applying it at some particular part of the pier, as a force or action tending to overturn it, an equilibrium is established between them. But this method will not do; because it is founded on the supposition that the constituent parts of the arch are perfectly polished, and at liberty to slide freely on each other. Whereas, on the contrary, the parts that compose the arch are completely hindered from sliding on each other, partly by their roughness and friction, and partly by the cement employed between them, and still more by the ties and fastenings placed within, to bind them together. By these means it happens, that all the parts are firmly compacted and united, so as to form the whole arch in some measure, into one rigid and solid mass; and besides that many of the voussoirs, in the lower parts of the arch, are built and bonded into the very body of the pier itself, and forming a part of its very mass.

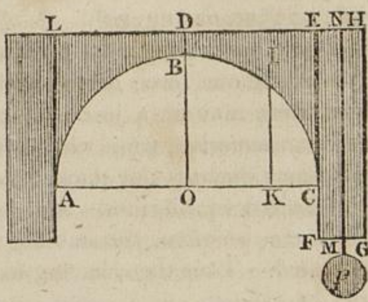
The same principle also, of the constant and determinate magnitude of the horizontal push, is founded on the supposition, that the arch is a true and real arch of equilibration; which perhaps can never be justly said to be the case. Besides, if it were such an arch, and the quantity of the constant horizontal push duly found, it would still be doubtful at what point of the pier to apply it, in making the calculation of its effect, on account of the circumstance that the arch has a bearing and oblique thrust, not against one point only, but in a different degree at all the points in that part of the pier extending from the impost, or foot of the arch, upward to the very top or roadway over the bridge.

On all these accounts then, and perhaps others, not here adverted to, it would seem that there is not, and perhaps cannot be, any true and perfect mathematical calculation made, of the exact balance between the push of an arch and the stability of the piers. Hence it has happened that various methods have been employed for this purpose, by different authors, with more or less show of reason or grounds of propriety: and hence also many practical engineers, neglecting all such calculations as unsatisfactory, have depended on practice and experience only, taking care, as they think, to err on the safe side, by making the piers much too massive, rather than risk the hazard of a failure by the chance of the contrary case. In this uncertainty, after several trials and examinations, two of the most promising, among the various ways of solving this problem, have been selected and delivered in the following prop. as affording probably a near approach to a true conclusion.

PROP. XIV.

To find the thickness of the piers of an arch, necessary to keep the arch in equilibrio, or to resist its drift or shoot, independent of any other arches.

First Solution.—Let BDEC be the half arch, and EFGH the pier necessary to balance and support it, considered as moveable about the extreme point G of the base.

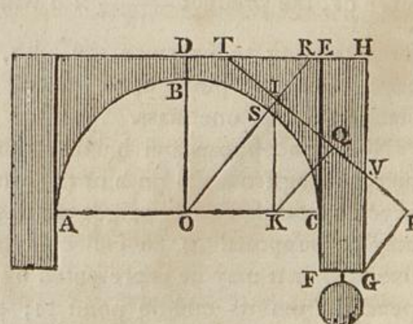


Through the centre of gravity I, of the arch BDEC, let IK be drawn perp. to the span AOKC. Now the semiarch BDEC is supported against the part of the pier EC, but chiefly on the impost or lowest point C, which sustains its weight, and by the horizontal thrust of the other semiarch ALDB, acting against it in the line of meeting BD. If both of these pressures be taken at their lowest points B, C, the arch may be considered as supported at these two points after the manner of a solid beam. But when such a body is supported in this way, it is well known, from the principles of mechanics, that the weight of the body downward, is in proportion to the horizontal push at its foot, as the vertical line IK is to the horizontal line KC; therefore the weight of the semiarch BDEC, is to its shoot against the pier at C, as IK is to KC: this force or push therefore will be expressed by $\frac{KC}{KI} \times a$, where a denotes the arch BDEC, or its weight or its area: and if this force be drawn into the length of the lever CF, the product $\frac{KC.CF}{IK} \times a$ will express the efficacious force tending to overturn the pier, by causing it to turn back about the point G, supposing the pier to be firmly compacted into one mass.

Now, to oppose and balance this force to overset the pier, arising from the push of the arch, we have the resistance depending on the weight of the pier itself. This weight may be supposed to be collected into its middle vertical line MN, or it may be represented by an equal weight p suspended from its middle point M; p, acting by the lever MG, and denoting the weight of the pier, or its area EF.FG, Therefore the resistance of the pier will be expressed by $EF . FG . \frac{1}{2}FG$ or $\frac{1}{2}EF . FG^2$.

Then, by making this opposing force of the pier equal to the efficacious force of the arch, both as expressed above, that there may be a just balance between them, they will form an equation, from which will easily be determined the unknown quantity, or thickness of the pier, so as to produce the desired equilibrium. And, by adding a little more to it, for better security, the stability is considered as sufficiently obtained. Thus then, having made the equation $\frac{1}{2}EF . FG^2 = \frac{KC . CF}{IK} . a$, its resolution gives us $FG = \sqrt{\frac{KC . CF}{IK . EF} . 2a}$, which is the first theorem or rule for the thickness of the pier; but which will probably be too small, by having taken the whole push of the arch as acting at the lowest point C.

Second Solution.—In the second mode of solving this problem, though the arch stones are supposed to be laid in mortar, and so cemented or locked together as to prevent them from easily sliding on one another, yet the whole not considered so firm or hard as to form as it were one solid stone; but the mortar or connection being only so firm, that if the piers were not sufficiently strong, the arch would break in the weakest part, and overturn the piers. In this method too let all the matter in the arch BDEC be supposed collected into its centre of gravity I, through which draw OI from the centre O, and through the joint SR of the arch in which the centre of gravity is situated: perpendicular to the joint SR draw IQP, the direction in which the



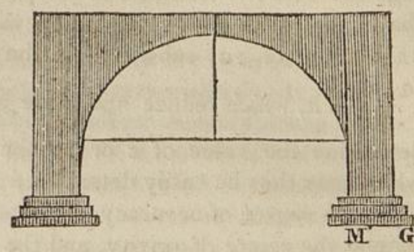
joint SR resists and supports the action of the arch at I: draw IK perpendicular to AC, or in the direction of gravity, also GP and KQ perpendicular to IP or parallel to OIR. Then if IK represent the weight of the arch BDEC in the direction of gravity, this will resolve into IQ the force acting against the pier perpendicular to the joint SR, and QK the part of the force parallel to the same: the line IQ is the only force acting perpendicular on the arm GP, of the crooked lever FGP, to turn the pier about the point G; consequently IQ × GP will express the efficacious force of the arch to overturn the pier, and which must be equal to the force of the pier itself, denoted by the area $EG \times \frac{1}{2}FG$ as before; that is $\frac{IQ}{IK} . a . GP = EF . FG . \frac{1}{2}FG = \frac{1}{2}EF . FG^2$, a denoting the area of the section BDEC of the arch, as EF . FG denotes the section EFGH of the pier. And this equation, after substituting for GP its value, will be a 2d theorem for the thickness of the pier, and which may probably be rather above the just quantity.

Schol.—As the centre of gravity is employed in both the preceding methods, it will be necessary to employ a few lines on the manner of finding the place I of that centre, together with the various other lines in the figure dependent on and connected with it. Now the centre of gravity I may be known either by mathematical calculation, or by mechanical and geometrical measurement. The best way of performing the first method seems to be on this principle, viz. 'That the content of the solid described by any plane surface, either in moving parallel to itself, or in revolving about a line as an axis, is always equal to the product of the generating plane, and the line described by its centre of gravity.' Hence, if the whole figure ODEC be first revolved about the axis OC, the rectangle ODEC will describe a cylinder, and the space OBSC, of a given figure, will describe a solid of a known magnitude; the difference of these two solids will give the content of the solid described by the mixed space BDECSB;

this solid content divided by the area of its said generating figure, gives the circumference of the circle described by the centre of gravity I, which circumference divided by the number 6·2832, or by $\frac{44}{7}$, will be the length of the radius IK. Next, by conceiving the same figure to revolve about the axis OD, and proceeding in the same way, there will be found the line OK, or the distance of the centre of gravity I from the axis OD. The point I being thus determined, there will hence be known all the lines KC, OI, RS, IQ, IT, TE, &c. Then, by denoting the unknown breadth of the pier, EH or FG, by any letter, as z, in terms of it will be expressed the perpendicular GP: thus, by similar triangles, as IK : OK :: TH : HV; hence GH − HV gives GV, and OI : IK :: GV : GP expresses the unknown line GP. Lastly, the value of GP substituted in the foregoing equation $\frac{1}{2}EF \cdot FG^2 = \frac{IQ \cdot GP}{IK} \cdot a$, it will be in the form of a quadratic, the solution of which will give the value of FG, the thickness of the pier sought, very near the truth.

The mechanical way of finding the centre of gravity I, and the geometrical measurement, is thus performed: On card-paper or pasteboard, or any other thin plate, construct the given figure BDECB very correctly, of a pretty large size, from a scale: then cut it out very neatly by the extreme edges, and lay it so as just to balance itself over the straight edge of a table, the line CE parallel to the edge, and close by the edge of the table draw a line on the paper, which will be the line IK; next balance the same figure in like manner with the line DE parallel to the edge of the table, close by which draw another line, crossing the former line in the point I, which will be the centre of gravity of the figure, determined sufficiently near the truth. This done, lay this point I down on another general construction of the figure, having the representation of the pier annexed, on which also draw all the other lines before mentioned, measuring their lengths by the scale of construction, and noting them down. Then with these, together with the thickness of the pier EH or FG, denoted by the unknown letter z, compute the value of GP, which, with z the value of FG, substitute in the equation $\frac{1}{2}\text{EF} \cdot \text{FG}^2 = \frac{\text{IQ} \cdot \text{GP}}{\text{IK}} \cdot a$, which reduce and solve as above mentioned, to determine the value of z or FG the thickness of the pier; which may thus be easily determined in all cases, and with a sufficient degree of accuracy.—The same methods of determining the centre of gravity, and the lines IK, KC, in the fig. to the 2d example following may be employed, to substitute in the expression $\text{FG} = \sqrt{\frac{\text{KC} \cdot \text{CF}}{\text{IK} \quad \text{EF}}} \cdot 2a$, for determining the thickness of the pier by the first rule.

In the foregoing solutions, it appears that, besides having given all the measures or dimensions of the arch and height of the pier, it is necessary to know the areas of their vertical transverse sections, or at least that of the superstructure BDEC: and this is easily to be found, when the figure of the arch BC and the exterior DE are known, viz, by deducting the area of the space or vacuity OBSC from that of the whole figure ODEC.—The foregoing solutions may also be considered as taking place either when the pier is all dry, or when it stands partly in water, which can penetrate its foundation or the joints of the masonry: and whether this last circumstance takes place or not, can probably be well judged of and ascertained by the experienced builder: if it do take place, which is perhaps commonly the case, then in the calculation the weight of the part in water must be reduced in the proportion of 5 to 3, as stone loses 2 parts in 5 of its weight when immersed in water.—In the foregoing solution it has also been supposed that the pier is made every where straight alike, or equally thick down to the very bottom, as represented in the two preceding figures. But, instead of that, it is very common to enlarge the pier towards the bottom, both to give it a broader base to stand on, without increasing the weight or dimensions above, and to make the lever WK longer at the base, to oppose a greater resistance to its oversetting or turning about the point G, and without any sensible increase to the weight of the pier. On the contrary, as the thickness, and consequently the weight of the pier, may be diminished above, in proportion as it is enlarged at the foundation, without diminishing its force of resistance and stability, the experienced architect will avail himself of the circumstance, to reduce in a considerable degree the size of the pier, and the expense of the work.



In the investigation of this proposition, the sections of the arch and pier are used for their solidities, as being evidently in the same proportion, or in that of their weights, since they are of the same length, viz, the breadth of the bridge. By the above rules then, the necessary thickness of a pier may be found, so that it shall *just* balance the spread or shoot of the arch, independent of any other arch on the side of the pier. But the weight of the pier ought a little to preponderate against, or exceed in effect, the shoot of the arch: and therefore the thickness ought to be taken a little more than what will be found by these rules; unless it be supposed that the pointed projections of the piers against the stream, beyond the common breadth of the bridge, will be a sufficient addition to the pier, to give it the necessary preponderancy. We may now take some examples of the calculation in numbers, to show the manner of operation, and in them also to point out the easiest methods of calculation.

EXAMPLE I.

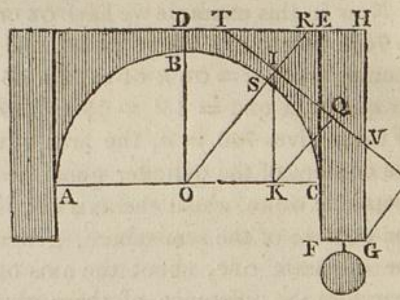
Supposing the arch in the figure to be a semicircle, whose height or pitch is 45 feet, and consequently its span 90 feet;

also supposing the thickness DB at top to be 7 feet, and the height CF to the springing 20; let it be required to find the thickness FG of the pier, necessary to resist the shoot of the arch; the roadway being a horizontal right line.

Now in this example we have OB or OC $= 45$, BD $= 7$, (fig. p. 74) OD or CE $= 52$, CF $= 20$, and EF $= 72$. Hence, the rectangle ODEC $=$ OD $\times$ OC $= 52 \times 45 = 2340$, and the circular quadrant OBC $= 45^2 \times \frac{11}{14} = 1590$ nearly, the difference of these gives $750 = a$, the area of the arch BDEC. Again, the content of the cylinder generated by the rotation of the rectangle ODEC, about the axis OD, is $4\text{OC}^2 \times \frac{11}{14} \times \text{OD}$; and the content of the semisphere, generated by the rotation of the quadrant OBC, about the axis OB, is $4\text{OC}^2 \times \frac{11}{14} \times \frac{2}{3}\text{OB}$; therefore the difference of these gives $4\text{OC}^2 \times \frac{11}{14} \times (\text{OD} - \frac{2}{3}\text{OB}) = 8100 \times \frac{11}{14} \times (52 - 30) = 8100 \times \frac{11}{14} \times 22 = 8100 \times \frac{11}{7} \times 11 = 140000$, for the content of the solid generated by the area BDEC (750) about the axis BD. Hence $140000 \div 750 = 186\frac{2}{3}$ the circumference or path described by the centre of gravity I about OD; consequently $186\frac{2}{3} \times \frac{7}{44} = 29{\cdot}7 =$ OK, the radius of that circle. Hence OC $-$ OK $= 45 - 29{\cdot}7 = 15{\cdot}3 =$ KC.

Again, the content of the cylinder generated by the rotation of the rectangle OBEC, about the axis OC, is $4\text{OD}^2 \times \frac{11}{14} \times \text{OC}$; and the content of the semisphere, as above, is $4\text{OB}^2 \times \frac{11}{14} \times \frac{2}{3}\text{OC}$; therefore the difference of these two $(\text{OD}^2 - \frac{2}{3}\text{OB}^2) \times \frac{44}{14} \times \text{OC}$, gives $(52^2 - \frac{2}{3}.45^2) \times \frac{22}{7} \times 45 = 1354 \times \frac{11}{7} \times 90 = 191494$, for the content of the solid generated by the area BDEC (750) about the axis OC. Hence $191494 \div 750 = 255{\cdot}325$ the circumference or path described by the centre of gravity I about OC; conseq. $255{\cdot}325 \times \frac{7}{44} = 40{\cdot}6 =$ IK, the radius of that circle. Lastly, the 1st theorem $\sqrt{\frac{\text{KC}.\text{CF}.2a}{\text{IK}.\text{EF}}}$ gives $\sqrt{\frac{15{\cdot}3 \times 20 \times 1500}{40{\cdot}6 \times 72}} = \sqrt{\frac{510000}{3248}} =$ $12\frac{1}{2}$ feet $=$ FG, for the required thickness of the pier; but which is probably below the truth, and perhaps below what a practical engineer would fully trust to.

It may be added, that the method of determining the place of the centre of gravity I, by balancing the figure BDEC, gave, within a small fraction, the same values of the two lines IK, KC, viz, 40 +, and 15 +, which were above calculated to be 40·6 and 15·3.



Secondly, to apply our example to the 2d theorem, $\frac{1}{2}$EF . FG^2 $= \frac{\text{IQ . GP}}{\text{IK}} \cdot a$, the same methods of determining the position of the centre of gravity I may be employed. If the mechanical method of balancing and measurement on a scale be used, we may then measure, not only the lines IK, OK, KC, but all the other lines also depending on it, as OI, OR, TI, TR, TE, KQ, IQ, &c, excepting only such lines as depend on the unknown breadth FG of the pier. But, instead of that, we shall calculate the accurate value of all the lines wanted by strict mathematical principles, as follows. In the example are given OB = OC = DE = 45, OD = CE = 32, CF = 20, EF = 72; and just above we have found by calculation OK = 29·7, KC = 15·3, IK = 40·6, and the area BDEC or a = 750 ; and we have to compute IQ and GP. Now OI = $\surd(\text{OK}^2 + \text{IK}^2)$ = $\surd(29{\cdot}7^2 + 40{\cdot}6^2)$ = 50·3 ; then by similar triangles OI : OK :: IK : IQ = 23·97.

Again, to get an expression for GP, put the required thickness of the pier EH or FG = z ; then, because by similar triangles, IK : OK :: OD : DR = 38·04,
and IK : IO :: OD : OR = 64·42, hence OR − OI = 14·12 = IR,
and OK : OI :: IR : TR = 23·91, also DE − DR = 6·96 = RE,
hence TR + RE = 30·87 = TE, and TH = TE + EH = TE + z,
then IK : OK :: TH : HV = 22·58 + 0·7315z,
and GH − HV = GV = 49·42 − ·7315z,
lastly OI : IK :: GV : GP = 39·89 − ·5904z.
These values being now substituted in the 2d theorem $\frac{1}{2}$EF .

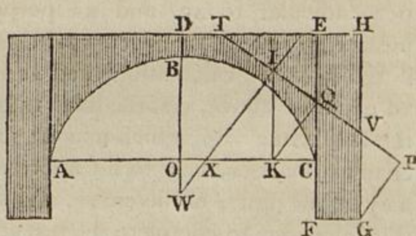
$FG^2 = \frac{IQ \cdot GP}{IK} \cdot a$, give $36z^2 = 17664\cdot9 - 261\cdot5z$, or $z^2 + 7\cdot26z = 490\cdot69$; the root of which quadratic equation gives $z = 18\cdot82 =$ EH or FG, the thickness of the pier sought.

It may be presumed that this theorem brings out the thickness of the piers very near the truth, and very near what would be allowed in practice by the best practical engineers, as may be gathered from a comparison of the two cases of Westminster and Blackfriars bridges, in the former of which the centre arch is a semicircle of 76 feet span, and 17 feet thickness of piers, and in the latter it is a semiellipse, of 100 feet span, 40 feet in height, and 19 feet thickness of piers.

EXAMPLE 2.

Suppose the span to be 100 feet, the height 40 feet, the thickness at top 6 feet, and the height of the pier to the springer 20 feet, as before.

Here the figure either is, or may be considered as, a scheme arch, or the segment of a circle, in which the versed sine OB is $= 40$, and the right sine OA or OC $= 50$; also DB $= 6$, CF $= 20$, and EF $= 66$. Now, by the nature of the circle, whose centre is W, the radius WB or WC $= \frac{OB^2 + OC^2}{2OB} = \frac{40^2 + 50^2}{80} = 51\frac{1}{4}$; hence OW $= 51\frac{1}{4} - 40 = 11\frac{1}{4}$; and the area of the semisegment OBC is found to be 1491;



which being taken from the rectangle ODEC $=$ OD $\times$ OC $= 50 \times 46 = 2300$, there remains $809 = a$, the area of the space BDECB. Hence, by the method of balancing this space, and measuring the lines, there will be found, KC $= 18$, IK $= 34\cdot6$, IX $= 42$, KX $= 24$, OX $= 8$, IQ $= 19\cdot4$, TE $= 35\cdot6$, and TH $= 35\cdot6 + z$, putting $z =$ EH, the breadth of the pier, as before. Then IK : KX : : TH : HV = $24 \cdot 7 + 0 \cdot 7z$; hence GH − HV = $41 \cdot 3 - 0 \cdot 7z$ = GV, and IX : IK :: GV : GP = $34 \cdot 02 - 0 \cdot 58z$. These values being now substituted in the theorem $\frac{1}{2}\text{EF} \cdot \text{FG}^2 = \frac{\text{IQ} \cdot \text{GP} \cdot a}{\text{IK}}$, give $33z^2 = 15431 \cdot 47 - 263 \cdot 09z$, or $z^2 + 8z = 467 \cdot 62$, the root of which quadratic equation gives $z = 18$ = EH or FG the breadth of the pier, and which it may be presumed is sufficiently near the truth.

These two cases it may be expected are sufficient to exemplify this method of determining the proper dimension of the piers; a method, the propriety of which is thus confirmed by conclusions that are so conformable to the practice of the best engineers. In all cases it appears to be the easiest course, and sufficiently correct, to construct accurately the semiarch and superstructure above it; then find its centre of gravity by the method of balancing it in two positions perpendicular to each other, viz. in lines parallel and perpendicular to the base AC; next through that centre I draw a line IW perpendicular to the curve of the arch, or in the direction of the arch joints there, and meeting the base line in the point X; next, through I draw TVP perpendicular to IX, and IK perpendicular to AC, and KQ perpendicular to TP. Then measure by the scale as many of these lines as are necessary in the intended calculation, and as are used in working the 2d example above, viz. the lines IK, KX, TE, IQ, and compute the area BDEC = a, which may be sufficiently done in a mechanical manner, and to an approximate degree, whatever may be the figure of the curve, and shape of that area. After this, continue to complete the rest of the calculation as in the example above.

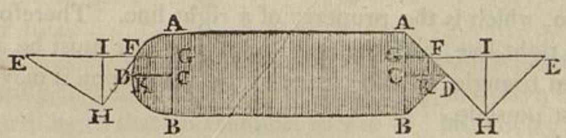
SECTION IV.

THE FORCE AND FALL OF THE WATER, &c.

PROP. XV.

To determine the Form of the Ends of a Pier, so as to make the Least Resistance, or be the Least subject, to the Force of the Stream of Water.

LET the following figure represent a horizontal section of the pier, AB its breadth, CD the given length or projection of the end, and ADB the line required, whether right or curved; also let EF represent the force of a particle of water acting on AD at the point F, in the direction parallel to the axis CD; produce EF to meet AB in G, and draw the tangent FH; also draw EH perpendicular to FH, HI perpendicular to EF, and FK perpendicular to DC.



Now the absolute force EF of the particle of water may be resolved into the two forces EH, HF, and in those directions; of these, the latter one, acting parallel to the face at F, is of no effect; and the former EH is resolved into the two EI, IH; so that EI is the only efficacious force of the particle to move the pier in the direction of its axis or length: That is, the absolute force is to the efficacious force, as EF is to EI. Then, since EF is the diameter of a semicircle passing through H, by the nature of the circle it will be, as $EF : EI :: EF^2 : EH^2 ::$ (by similar triangles) $HF^2 : HI^2$ and :: the square of the fluxion of the curve or line : the square of the fluxion of the ordinate FK, because HF, HI are parallel to the line and ordinate.

Therefore, putting the abscissa DK $= x$, the ordinate KF $= y$, and the line DF $= z$, it will be, as $\dot{z}^2 : \dot{y}^2 :: 1$ (the force EF) : $\frac{\dot{y}^2}{\dot{z}^2} =$ the force of the particle at F to move the pier in the direction EFG. But the number of particles striking against the indefinitely small part of the line, is as $\dot{y}$; this drawn into the above found force of each, we have $\frac{\dot{y}^3}{\dot{z}^2} = \frac{\dot{y}^3}{\dot{x}^2 + \dot{y}^2}$ for the fluxion of the force, or the force acting against the small part $\dot{z}$ of the line.

But, by the proposition, the whole force on DFA must be a minimum, or the fluent of $\frac{\dot{y}^3}{\dot{x}^2 + \dot{y}^2}$ must be a minimum, when that of $\dot{x}$ becomes equal to the constant quantity DC; in which case it is known that $\frac{\dot{x}\dot{y}^3}{(\dot{x}^2 + \dot{y}^2)^2}$ must be always equal to some constant quantity q; and hence $\dot{x}\dot{y}^3 = q \times (\dot{x}^2 + \dot{y}^2)^2$.

Now, in this equation, it is evident that $\dot{x}$ is to $\dot{y}$ in a constant ratio; but when two fluxions are always in a constant ratio, their fluents x, y, it is known, are also in a constant ratio, which is the property of a right line. Therefore DFA is a right line, and the end ADB of the pier must be a right-lined triangle, that the force of the water upon it may be the least possible.

PROP. XVI.

To determine the Quantity of the Resistance of the End of a Pier against the Stream of Water.

Using here the same figure and notation as in the last proposition, by the same it is found, that the fluxion of the force of the stream against the face DF, is $\frac{\dot{y}^3}{\dot{x}^2 + \dot{y}^2}$; and since the fluxion of the force against the base is $\dot{y}$, it follows, that the force of the stream against the base AB, is to the force against the face ADB, as (y) the fluent of $\dot{y}$, is to the fluent of $\frac{\dot{y}^3}{\dot{x}^2 + \dot{y}^2}$. That is, the absolute force of the stream, is to the efficacious force against the face of the pier, as its breadth is to double the fluent of $\frac{\dot{y}^3}{\dot{x}^2 + \dot{y}^2}$, when y is equal to half the breadth.

Corollary 1.—If the face ADB be rectilineal.

Putting DC $= a$, AC $= b$, and AD $= \sqrt{(aa + bb)} = c$; then, as $a : b :: x : y$ by similar triangles; hence $x = \frac{ay}{b}$, and $\dot{x} = \frac{a\dot{y}}{b}$; this being written for it in the general expression above, it becomes $\frac{bb\dot{y}}{aa + bb} = \frac{bb\dot{y}}{cc}$, for the fluxion of the force on AD; the fluent of which, or $\frac{bby}{cc}$, is the force itself. Consequently the force on the flat base AB, is to that on the triangular end ADB, as y to $\frac{bby}{cc}$, or as cc to bb, that is, as AD^2 to AC^2.

And if AC be equal to CD, or ADB a right angle, which is generally the case, then $AD^2 = 2AC^2$, and the force on the base will be to that on the face, as 2 to 1. Moreover, as the force on ADB, when ADB is a right angle, is only half of the absolute force, so it is evident that the force will be more than one-half when ADB is greater than a right angle, and less when it is less; and also, that the longer AD is, the less the force is, it being always inversely as the square of AD.

Corollary 2.—If ADB be a semicircle.

The radius AC $=$ CD $= a$; then $2ax - xx = yy$, or $x = a - \sqrt{(aa - yy)}$, ard $= \frac{y\dot{y}}{(\sqrt{aa - yy})}$; hence $\frac{\dot{y}^3}{\dot{x}^2 + \dot{y}^2}$ becomes

$\frac{aa - yy}{aa} \times \dot{y}$, the fluent of which is $\frac{aa - \frac{1}{3}yy}{aa} \times y$; and therefore the force on the base is to the force on the circular end, as y is to $\frac{aa - \frac{1}{3}yy}{aa} \times y$, or as aa to $aa - \frac{1}{3}yy$, or as $3aa$ to $3aa - yy$. And when $y = a =$ AC, the proportion becomes that of 3 to 2. So that, only one-third of the absolute force is taken off by making the end a semicircle.

Corollary 3.—When the face ADE is a parabola.

Then, the notation being as before, viz, DC $= a$, and AC $= b$, it is $a : x :: bb : yy$; hence $x = \frac{ayy}{bb}$, and $\dot{x} = \frac{2ay\dot{y}}{bb}$; which being written in the general expression, the fluent of it becomes the circular arc whose radius is $\frac{bb}{2a}$ and tangent y, or $= \frac{bb}{2a} \times$ arc whose radius is 1 and tangent $\frac{2ay}{bb}$; so that the absolute force is to the force on the parabolic end, as y is to the arc whose tangent is y and radius $\frac{bb}{2a}$; that is, as the tangent of an arc is to the arc itself, the radius being to the tangent, as 1 to $\frac{2ay}{bb}$, or as 2 to $\frac{ay}{bb}$. And when $y = b$, the ratio of the tangent to radius, is that of 2 to $\frac{b}{a}$; or that of 2 to 1 when DC $=$ CA. In which case, the whole force is to the force on the parabolic end, as the tangent, which is double the radius, is to the corresponding arc; that is, as the tangent of 63° 26′ 4″ to the arc of the same, or as 2 to 1·10714; which is a less force than on the circle, but greater than on the triangle. And so on for other curves; in which it will be found, that the nearer they approach to right lines, the less the force will be, and that it is least of all in the triangle, in which it is one-half of the whole absolute force when right-angled.

It must be noted, however, that in determining the best form of the end of the pier to be a right-lined triangle, the water is supposed to strike every part of it with the same velocity: had the variably increased velocity been used, the form of the ends would come out a little curved; but as the increase of the velocity in the best bridges is very small, the difference in them is quite imperceptible.

PROP. XVII.

To determine the Fall of the Water in the Arches.

Having, in the foregoing propositions, treated of the resistance made by the piers to the current of water, it will now be proper to contemplate the effects of that resistance, and of the contraction of the passage they produce in the water-way. These effects are, a fall, or sudden steep descent, and an increase of velocity in the stream of water, just under the arches, more or less in proportion to the quantity of the obstruction; being somewhat observable at the place of all bridges, even where the arches are very large and the piers small, but in a high and extraordinary degree at London bridge, and some others, where the piers, and the sterlings, are so very large, in proportion to the arches. Now, in an open canal or river, an equal quantity of water passing in every part, in the same time, if in any part the passage be narrower, there, the bottom continuing the same, the velocity of the stream must be so much the greater, and a correspondent rise in the surface must also take place, to produce that increased celerity. Similar effects also occur in a river when any obstacles, as the piers of a bridge, are placed in the way of a stream. This is resisted and obstructed by the piers; of course the water rises against them, and consequently the stream from thence descends the more rapidly. And this is the case, not only in such canals or rivers where the stream runs always the same way, but in tide rivers also, both upward and downward. During the time of flood, when the tide is flowing upward, the rise of the water is against the under side of the piers; but the difference between the two sides gradually diminishes as the tide flows less rapidly towards the conclusion of the flood. When this has attained its full height, and there is no longer any current, but a stillness prevails in the water for a short time, the surface assumes an equal level, both above and below bridge. But, as soon as the tide begins to ebb again, the resistance of the piers against the stream, and the contraction of the water-way, cause a rise of the surface above and under the arches, with a fall and a more rapid descent in the contracted stream just below. The quantity of this rise, and of the consequent velocity below, keep both gradually increasing, as the tide continues ebbing, till at quite low water, when the stream or natural current being the quickest, the fall below the arches is the greatest. And it is the quantity of this fall which it is the object of this problem to determine.

Now, the motion of free running water is the consequence of, and produced by the force of gravity, as well as that of any other falling body. Hence the height due to the velocity, that is, the height to be freely fallen by any body to acquire the observed velocity of the natural stream, in the river a little above the bridge, becomes known. From the same velocity also will be found that of the increased stream in the narrowed way of the arches, by taking it in the reciprocal proportion of the breadth of the river above, to the contracted way in the arches; viz. by saying, as the latter is to the former, so is the first velocity, or slower motion, to the quicker. Next, from this last velocity, will be found the height due to it as before, that is, the height to be freely fallen through by gravity, to produce it. Then the difference of these two heights, thus freely fallen by gravity, to produce the two velocities, is the required quantity of the water-fall in the arches; allowing however, in the calculation, for the contraction of the stream, in the narrowed passage, at the rate as observed by Sir I. Newton. Such then are the elements and principles on which the solution of the problem is to be made out; and which it is now easy for any one to perform.

But, as it may be desirable to exhibit the manner of the solution of this curious problem, by some former noted authors, in this instance I shall give the solution from some manuscripts that have now been many years in my possession: viz, one solution by the celebrated Wm. Jones, Esq. the friend of Sir I. Newton, and father of the late Sir Wm. Jones; which is in Mr. Jones's own hand writing, and which I had from the late Mr. John Robertson, many years clerk and librarian to the Royal Society, who had the paper from Mr. Jones himself. Another solution is by the same Mr. Robertson himself, from a paper found among a great number of other manuscripts which I purchased at the sale of his books, after his death in the year 1776; and among which papers there are also other solutions that have never been published. The solutions here inserted, are given in the same words and peculiar manner as in those authors, in order to show their different forms and modes of stating and working. And first the solution by Mr. Jones, done in his usual manner, which was always remarkably concise, neat, and accurate.

The Solution of Wm. Jones, Esq.

" *Lemma.* In a chanel, whose stream runs with such an uniform velocity, in any given time, as is acquired by falling from a certain hight (h); if an obstacle should contract the passage of the water, in any place, the water above the obstacle will rise to such a hight (H) as to acquire a velocity that will discharge the stream as it comes; but will occasion a fall at the obstacle: and the difference ($H - h$) between these hights, is the measure of that fall.

" In a chanel of running water, whose breadth (b feet), and the velocity of its stream (v feet in 1″), being given: To determine the quantity of the fall, occasioned by an obstacle that takes up p feet of the breadth of the chanel.

" Let the hight fallen (near the surface of the earth) in 1″

of time, be (a feet); and the contraction of streams, in the water-way, be as r to 1. Put $c = \frac{b}{b-p}$; $d = rrcc$: Then the quantity of the fall is $\overline{d-1} \times vv \times \frac{1}{4a}$ feet.

"For, the water-way takes up w $(\frac{b-p}{b})$ part of the breadth of the chanel. But streams are found to be contracted in the water-way, in the proportion of r to 1. Therefore the water-way contracted will be $(\frac{w}{r} =)\ \frac{1}{rc}\ (= m)$. But the current above the obstacle moves v feet in 1″ of time; and the velocities of water through different passages, of the same hight, are as the reciprocals of the breadth of those passages. Therefore the current, in the true water-way, must move $(\frac{v}{m} = \frac{1}{m}v =)\ nv$ feet in 1″ of time.

"Now, since (a) feet is the hight fallen in 1″ of time to acquire a velocity to move uniformly the length $2a$ in that time: Let x and z feet be the hights fallen to acquire a velocity to move uniformly the lengths v and nv feet in 1″ of time: and because hights fallen are as the squares of their velocities; therefore $\frac{\overline{2a}|^2}{vv} = \frac{a}{x}$, and $\frac{\overline{2a}|^2}{nnvv} = \frac{a}{z}$: consequently $x = \frac{vv}{4a}$, and $z = \frac{nnvv}{4a}$. That is, $\frac{vv}{4a}$ feet is the hight of water necessary to produce, in the chanel, a current that moves v feet in 1″ of time. And $\frac{nnvv}{4a}$ feet, is the hight of water necessary to produce, in the water-way, a current that moves nv in that time. Then the difference $\overline{nn-1} \times \frac{vv}{4a}$ of these hights, is the fall in feet. But $n = (\frac{1}{m} =)\ rc$, therefore $nn = rrcc = d$ per supposition. Therefore $\overline{d-1} \times vv \times \frac{1}{4a}$ feet, is the quantity of the fall. Q. E. D.

" Hence, putting $\text{A} = \text{L}.\frac{1}{4a}$, $\text{B} = \text{L}.r$, $\text{C} = \text{L}.c$, $\text{D} = 2 \times \overline{\text{B} + \text{C}} = \text{L}.d$: Then $\text{L}.\overline{d - 1} + 2\text{L}.v + \text{A} =$ Log. of the quantity of the fall, in feet*.

" Now, if the length of a pendum vibrating seconds, is 39·126 inches, then will $a = 16{\cdot}0899$ feet; and, according to Newton, $r = \frac{25}{21}$: consequently $\text{A} = \bar{2}.1913861$; and $\text{B} = 0.0757207$."

Such is the solution of this problem as given by Mr. Jones. And as there is contained in the same paper with this, a short solution of another kindred problem, it is here inserted, as follows.

" The length, p inches, of a pendulum that performs one vibration in 1″ of time, at a given place, being known; the altitude (a) fallen from, in 1″ of time, will be $\frac{1}{2}p\pi\pi$ inches, or $\frac{1}{24}p\pi\pi$ feet, at that place.

" For $\left(\frac{\text{time of } 1''}{\text{time in } \frac{1}{2}p} = \right) \frac{\tau}{t} = \frac{c}{d} = \frac{\pi\pi}{1}$; therefore $\left(\frac{\tau\tau}{tt} = \right) \frac{a}{\frac{1}{2}p} = \left(\frac{cc}{dd} = \right) \frac{\pi\pi}{1}$.

" Consequently $a = \frac{1}{2}p\pi\pi$ inches $= \frac{1}{24}p\pi\pi$ feet.

" And putting $\text{N} = (\text{L}.\frac{1}{24}\pi\pi = 2\text{L}.\pi - \text{L}.24 = \bar{1}.6140885$; then $\text{L}.a = \text{L}.p + \text{N}$."

Proceed we now to Mr. Robertson's solution of the problem, which is on the principles, but more in detail, than Mr. Jones's. This solution was published by Mr. R. in the Philos. Trans. vol. 50, or in my new Abridgement, vol. 13, from which it is chiefly here extracted.

Mr. John Robertson's Solution of the Problem.

" Sometime before the year 1740, the problem about the fall of water, occasioned by bridges built across a river, was

* This is the theorem, adapted to working by logarithms, given by Mr. Jones to Mr. Gardiner, and printed in p. 12 of his Logarithms in 4to; the latter L denoting logarithm, in the theorem.

much spoken of at London, on account of the fall that was supposed would be at the new bridge to be built at Westminster. In Mr. Hawksmoor's and Mr. Labelye's pamphlets, the former published in 1736, and the latter in 1739, the result of Mr. Labelye's computations was given: but neither the investigation of the problem, nor any rules, were at that time published.

"In the year 1742 was published, Gardiner's edition of Vlacq's Tables; in which, among the examples there prefixed, to show some of the uses of those tables, drawn up by the late Wm. Jones, esq. there are two examples, one showing how to compute the fall of water at London-bridge, and the other applied to Westminster-bridge: but that excellent mathematician's investigation, by which those examples were wrought, was not printed, though he communicated copies of it to several of his friends. Since that time, it seems as if the problem had in general been forgotten, as it has not made its appearance, to my knowledge, in any of the subsequent publications. As it is a problem somewhat curious, though not difficult, and its solution not generally known, (having seen four different solutions, one of them very imperfect, extracted from the private books of an office in one of the departments of engineering in a neighbouring nation), I thought it might give some entertainment to the curious in these matters, if the whole process were published.

"PRINCIPLES.

"1. A heavy body, that in the first second of time has fallen the height of a feet, has acquired such a velocity, that, moving uniformly with it, will in the next second of time move the length of $2a$ feet.

"2. The spaces run through by falling bodies are proportional to one another as the squares of their last or acquired velocities.—These two principles are demonstrated by the writers on mechanics.

"3. Water forced out of a larger chanel, through one or more smaller passages, will have the streams through those passages contracted in the ratio of 25 to 21.—This is shown in the 36th prob. of the 2d book of Newton's Principia.

" 4. In any stream of water, the velocity is such, as would be acquired by the fall of a body from a height above the surface of that stream.—This is evident from the nature of motion.

" 5. The velocities of water through different passages of the same height, are reciprocally proportional to their breadths.—For, at some time, the water must be delivered as fast as it comes; otherwise the bounds would be overflowed. At that time, the same quantity, which in any time flows through a section in the open chanel, is delivered in equal time through the narrower passages; or the momentum in the narrow passages must be equal to the momentum in the open chanel; or the rectangle under the section of the narrow passages, by their mean velocity, must be equal to the rectangle under the section of the open chanel by its mean velocity. Therefore the velocity in the open chanel is to the velocity in the narrower passages, as the section of those passages is to the section of the open chanel. But, the heights in both sections being equal, the sections are directly as the breadths. Consequently the velocities are reciprocally as the breadths.

" 6. In a running stream, the water above any obstacles put therein, will rise to such a height, that by its fall the stream may be discharged as fast as it comes.—For the same body of water, which flowed in the open chanel, must pass through the passages made by the obstacles: and the narrower the passages, the swifter will be the velocity of the water: but the swifter the velocity of the water, the greater is the height, from which it has descended: consequently the obstacles, which contract the chanel, cause the water to rise against them. But the rise will cease, when the water can run off as fast as it comes: and this must happen when, by the fall between the obstacles, the water will acquire a velocity in a reciprocal proportion to that in the open chanel, as the breadth of the open chanel is to the breadth of the narrow passages.

" 7. The quantity of the fall, caused by an obstacle in a running stream, is measured by the difference between the heights fallen from, to acquire the velocities in the narrow passages and open chanel.—For, just above the fall the velocity of the stream is such, as would be acquired by a body falling from a height higher than the surface of the water: and at the fall, the velocity of the stream is such, as would be acquired by the fall of a heavy body from a height more elevated than the top of the falling stream; and consequently the real fall is less than this height. Now as the stream comes to the fall with a velocity belonging to a fall above its surface; consequently the height belonging to the velocity at the fall, must be diminished by the height belonging to the velocity with which the stream arrives at the fall.

" PROBLEM.

" In a chanel of running water, whose breadth is contracted by one or more obstacles; the breadth of the chanel, the mean velocity of the whole stream, and the breadth of the water-way between the obstacles, being given; to find the quantity of the fall occasioned by those obstacles.

" Let b = breadth of the chanel in feet;
v = mean velocity of the water in feet per second;
c = breadth of the water-way between the obstacles.

Now $25 : 21 :: c : \frac{21}{25}c$, the water-way contracted, by prin. 3.

And $\frac{21}{25}c : b :: v : \frac{25b}{21c}v$, the veloc. in the contr. way, prin. 5.

Also $(2a)^2 : vv :: a : \frac{vv}{4a}$, height fallen to gain the vel. v, 1 and 2.

And $(2a)^2 : (\frac{25b}{21c}v)^2 :: a : (\frac{25b}{21c})^2 \times \frac{vv}{4a}$, ditto for the velocity $\frac{25b}{21c}v$, by princ. 1 and 2.

Then $\frac{25b}{21c} \times \frac{vv}{4a} - \frac{vv}{4a}$ is the measure of the fall required, prin. 7.

Or $[(\frac{25b}{21c})^2 - 1] \times \frac{vv}{4a}$ is a rule for computing the fall.

Here $a = 16{,}0899$ feet; and $4a = 64{,}3596$.

" EXAMPLE 1. *For London-Bridge.*

" By the observations made by Mr. Labelye in 1746,
The breadth of the Thames at London-bridge is 926 feet;
Sum of water-ways at the time of low water is 236 feet;
Mean veloc. of stream just above bridge is $3\frac{1}{6}$ f. per sec.
Under almost all the arches there are great numbers of dripshot piles, or piles driven into the bed of the water-way, to prevent it from being washed away by the fall. These dripshot piles considerably contract the water-ways, at least $\frac{1}{6}$ of their measured breadth, or about $39\frac{1}{3}$ feet in the whole. So that the water-way will be reduced to $196\frac{2}{3}$ feet.

" Now $b = 926$; $c = 196\frac{2}{3}$; $v = 3\frac{1}{6}$; $4a = 64{,}3596$.

Then $\frac{25b}{21c} = \frac{23150}{4130} = 5{,}60532$; its square $= 31{,}4196$;

And $31{,}4196 - 1 = 30{,}4196 = (\frac{25b}{21c})^2 - 1$;

Also $vv = (\frac{19}{6})^2 = \frac{361}{36}$; And $\frac{vv}{4a} = \frac{361}{36 \times 64{,}3596} = 0{,}15581$.

Then $30{,}4196 \times 0{,}15581 = 4{,}739$ f. $= 4$ f. $8{,}868$ inc. the fall required.

" By the most exact observations made about the year 1736, the measure of the fall was 4 feet 9 inches."

" EXAMPLE 2. *For Westminster-Bridge.*

" Though the breadth of the river at Westminster-bridge is 1220 feet; yet, at the time of the greatest fall, there is water through only the 13 large arches, which amount to 820 feet: to which adding the breadth of the 12 intermediate piers, equal to 174 feet, gives 994 for the breadth of the river at that time; and the velocity of the water just above the bridge, from many experiments, is not greater than $2\frac{1}{4}$ feet per second.

"Here $b = 994$; $c = 820$; $v = 2\frac{1}{4}$; $4a = 64,3596$.

Now $\frac{25b}{21c} = \frac{24850}{17220} = 1,443$; and its square $= 2,082$;

Hence $2,082 - 1 = 1,082 = (\frac{25b}{21c})^2 - 1$.

Also $vv = (\frac{9}{4})^2 = \frac{81}{16}$; and $\frac{vv}{4a} = \frac{81}{16 \times 64,3596} = 0,0786$.

Then $1,082 \times 0,0786 = 0,084$ f. $= 1$ inch, the fall required; and is about half an inch more than the greatest fall observed by Mr. Labelye."

Among the old papers of Mr. Robertson I find several other solutions of the same problem, by different persons, and on somewhat different principles. Several of the papers also, which are of a miscellaneous nature, relate to other branches of the subject of bridges; some of which, being curious, I shall avail myself of, by insertion in the appendix to this Tract.—The following table shows, at one view, the quantity of fall in the water under the arches, in consequence of its obstruction and contraction by the piers, according to several rates of velocity and quantity of obstacles; as computed on the foregoing principles.

A Table of the natural Rise of Water, in Proportion to the Resistance or Obstruction it meets with, in its Passage.

Construction of a modern bridge of 2 arches.	Velocity of the Current in one Second.	OBSTRUCTIONS, OR RESISTANCES.							Stages of Accumulation in Floods.	Construction of an ancient bridge of 3 or more arches.
Resistance 1-11th		1-8th	1-4th	3-8ths	1-half	5-8ths	3-4ths	7-8ths		Resistance 5-18ths.
Rise of Water.		Proportional Rise of Water, in Feet, Inches, and Parts.								Rise of Water.
ft. in. pts.		ft. in. pts.	ft. in. pts.	ft. in. pts.	ft. in. pts.	ft. in. pts.	ft. in. pts.	ft. in. pts.		ft. in. pts.
0 0 ·133	1 foot	0 0 ·158	0 0 ·283	0 0 ·49	0 0 ·87	0 1 ·69	0 4 ·041	1 4 ·728	Uniform Tenors.	0 0 ·320
0 0 ·533	2 feet	0 0 ·635	0 1 ·133	0 1 ·96	0 3 ·48	0 6 ·77	1 4 ·164	5 6 ·9		0 1 ·28
0 1 ·2	3 feet	0 1 ·428	0 2 ·549	0 4 ·41	0 7 ·835	1 3 ·234	3 0 ·368	12 6 ·53	Ordinary Floods.	0 2 ·881
0 2 ·133	4 feet	0 2 ·539	0 5 ·439	0 7 ·89	1 1 ·928	2 3 ·08	5 4 ·656	22 3 ·6		0 5 ·119
0 3 ·333	5 feet	0 3 ·967	0 7 ·083	1 0 ·25	1 9 ·763	3 6 ·316	8 5 ·024	34 10 ·31	Extra-ordinary Floods.	0 8 ·003
0 4 ·799	6 feet	0 5 ·713	0 10 ·199	1 5 ·64	2 7 ·339	5 0 ·934	12 1 ·476	50 2 ·112		0 11 ·525
Piers 12 / River 132	Velocities above seldom happen.	Piers 20 / Arches 140	Piers 40 / Arches 140	Piers 60 / Arches 160	Piers 80 / Arches 160	Piers 100 / Arches 160	Piers 120 / Arches 160	Piers 140 / Arches 160	Torrents above generally Inundations.	Piers 50 / River 180
This, next to one arch which has no resistance, without the flood encroaches on its crown, the most eligible mode.	N. B. These several numbers, respectively, show how high the water is constrained to rise above its natural level, or surface; which would otherwise carry it off, in a free and uninterrupted passage; therefore these numbers must everywhere be added to the depth of water, below the fall, to give the true height of the flood.—The seven predicaments above show the excellence or imperfection of bridges of every construction, and in all states of a flood, either in its uniform or variable tenors; and by which appears the great advantage of bridges of a sufficient capacity, and the pernicious consequences of all such as are not so.—London-bridge is nearly in the 6th predicament of this table, and Westminster-bridge nearly in the 2d. At the 1st of these the Thames, with a velocity of about 3f. 2in. per second, rises to about 4f. 7in. and at the latter, with a velocity of 2f. 6in. per second, to only 2 inches and a half.									In this most common mode, seldom sufficient in a flood, the water soon encroaches on the arches, and changes the predicament.

SECTION V.

OF THE TERMS OR NAMES OF THE VARIOUS PARTS PECULIAR TO A BRIDGE, AND THE MACHINES, &C, USED ABOUT IT; DISPOSED IN ALPHABETICAL ORDER.

Abutment, or Butment, which see in its place below.

Arch, an opening of a bridge, through or under which the water and vessels pass, and which is usually supported by piers or by butments. Arches are denominated circular, elliptical, cycloidal, catenarian, &c, according to the figure of the curve of them. There are also other denominations of circular arches, according to the different parts of a circle: So, a semicircular arch, is half the circle; a scheme or skeen arch, is a segment less than the semicircle; and arches of the third and fourth point, or gothic arches, consist of two circular arcs, excentric and meeting in an angle at top, each being 1-3d or 1-4th, &c, of the whole circle.

The chief properties of the most considerable arches, with regard to the extrados they require, &c, may be learned from the second section. It there appears, that none, but the arch of equilibration in the 2d example to prop. 5, can admit of a horizontal line at top: that this arch is not only of a graceful, but of a convenient form, as it may be made higher or lower at pleasure with the same opening: that, with a horizontal top, it can be equally strong in all its parts, and therefore ought to be used in all works of much consequence. All the other arches require tops that are curved, either upward or downward, some more and some less. Of these, the elliptical, or the cycloidal arch, seems to be the fittest to be substituted instead of the balanced one, with the least degree of impropriety: it is in general also the best form for most bridges, as it can be made of any height to the same span, or of any span to the same height, while at the same time its flanks are sufficiently elevated above the water, even when it is pretty flat at top; a property of which the other curves are not possessed in an equal degree: and this property is the more valuable, because it is remarked that, after any arch is built, and the centering struck, it settles more about the hanches than the other parts, by which other curves are reduced near to a straight line at the flanks. Elliptical arches also look bolder, are really stronger, and require less materials and labour than the others. Of the other curves, the cycloidal arch is next in quality to the elliptical one, for all the above properties. And, lastly, the circle. As to the others, the parabola, hyperbola, and catenary, they may not at all be admitted in bridges of several arches; but may in some cases be used for a bridge of one single arch, which is to rise very high, because then not much loaded at the flanks. We may hence also perceive the fallacy of those arguments which assert, that because the catenarian curve supports itself equally in all its parts, it will therefore best support any additional weight laid upon it: for the additional building made to raise the bridge to a horizontal line, or nearly such, by pressing more in one part than another, must force those parts down, and the whole must fall. Whereas, other curves will not support themselves at all, without some additional parts built above them, to balance them, or to reduce their parts to an equilibrium.

ARCHIVOLT, the curve or line formed by the upper sides of the voussoirs or arch stones. It is parallel to the intrados or underside of the arch when the voussoirs are all of the same length; otherwise not. By the archivolt is also sometimes understood the whole set of voussoirs.

BANQUET, the raised foot path at the sides of the bridge next the parapet. This ought to be allowed in all bridges of any considerable size: it should be raised about a foot above the middle or horse passage, being made 3, 4, 5, 6, 7, &c, feet broad, according to the size of the bridge, and paved with large stones, of a length equal to the breadth of the walk.

Battardeau, or Coffer-dam, a case of piling, &c, without a bottom, fixed in the bed of the river, water-tight or nearly so, by which to lay the bottom dry for a space large enough to build the pier on. When it is fixed, its sides reaching above the level of the water, the water is pumped out of it, or drawn off by engines, till the included space be laid dry; and it is kept so, by the same means, if there are leaks which cannot be stopped, till the pier is built up in it; and then the materials of it are drawn up again.

Battardeaux are made in various manners, either by a single inclosure, or by a double one, with clay or chalk rammed in between the two, to prevent the water from coming through the sides. And these inclosures are also made, either with piles only, driven close by one another, and sometimes notched or dove-tailed into each other; or with piles, grooved in the sides, and driven in at a distance from one another, with boards let down between them in the grooves.

The method of building in battardeaux cannot well be used where the river is either deep or rapid. It also requires a very good natural bottom of solid earth or clay: for, though the sides be made water-tight, if the bottom or bed of the river be of a loose consistence, the water will ooze up through it, in too great abundance to be evacuated by the engines. It is almost needless to remark, that the sides must be made very strong, and well propt or braced on the inside, to prevent the ambient water from pressing the sides in, and forcing its way into the battardeaux.

Bridge, a work of carpentry, masonry, or iron, built over a river, canal, &c, for the conveniency of crossing the same. A bridge is an edifice forming a way over a river, &c, supported by one arch, or by several arches, and these again supported by proper piers or butments. A stately bridge, over a large river, is one of the most noble and striking pieces of human art. To behold huge and bold arches, composed of an immense quantity of small materials, as stones, bricks, &c, so disposed and united together, that they seem to form but one solid compact body, affording a safe passage for men and carriages over large waters, which with their navigation pass free and easy under them at the same time, is a sight truly surprizing and affecting.

To the absolutely necessary parts of a bridge, already mentioned, viz, the arches, piers, and abutments, may be added the paving at top, the parapet wall, either with or without a balustrade, &c; also the banquet, or raised foot way, on each side, leaving a sufficient breadth in the middle for horses and carriages. The breadth of a bridge for a great city should be such as to allow an easy passage for three carriages and two horsemen a-breast in the middle way, and for three foot passengers in the same manner on each banquet. And for other less bridges, a less breadth.

As a bridge is made for a way or passage over a river, &c, so it ought to be made of such a height, as will be quite convenient for that passage; but yet so as to be consistent with the interest and concerns of the river itself, easily admitting through its arches the craft that navigate on it, and all the water, even at high tides and floods. The neglect of this precept has been the ruin of many bridges, and particularly that at Newcastle, over the river Tyne, on the 17th of November 1771. So that, in determining its height, the conveniences both of the passage over it, and under it, should be considered, and the height made to answer the best for them both, observing to make the *convenient* give place to the *necessary*, when their interests are opposite.

Bridges are generally placed in a direction perpendicular to the stream in a direct line, to give free passage to the water, &c. But some think they should be made, not in a straight line, but convex towards the stream, the better to resist floods, &c. And some such bridges have been really made.—Again, a bridge should not be made in too narrow a part of a navigable river, or one subject to tides or floods: because the breadth being still more contracted by the piers, will increase the depth, velocity, and fall of the water under the arches, and endanger the whole bridge and navigation.

Bridges are usually made with an odd number of arches, as one, or three, or five, or seven, &c; either that the middle of the stream or chief current may flow freely without the interruption of a pier; or that the two halves of the bridge, by gradually rising from the ends to the middle, may there meet in the highest and largest arch; or else, for the sake of grace, that by being open in the middle, the eye, in viewing it, may look directly through there, as one always expects to do in looking at it, and without which opening we generally feel a disappointment in viewing it.

If the bridge be equally high throughout, the arches, being all of a height, are made all of a size; which causes a great saving of centring. If the bridge be higher in the middle than at the ends, the arches are made to decrease from the middle towards each end, but so, as that each half may have the arches exactly alike, and that they decrease in span, proportionally to their height, so as to be always the same kind of figure, and similar parts of that figure: thus, if one be a semicircle, the rest should be semicircles also, but proportionally less; if one be a segment of a circle, the rest should be similar segments of other circles; and so for other figures. The arches being equal at equal distances, on both sides of the middle, is not only for the strength and beauty of the bridge, but that the centring of the one half may serve for the other also. But if the bridge be higher at the ends than the middle, which is a very uncommon case, the arches ought to increase in span and pitch from the middle towards the ends. When the middle and ends are of different heights, their difference however ought not to be great in proportion to the length, that the ascent may be easy; and then also it is more beautiful to make the top one continued curve, like Blackfriars, than two inclined straight lines, from the ends towards the middle, like that of Westminster bridge.

Bridges should rather be of few and large arches, than of many and small ones, if the height and situation will allow of it; for this will leave more free passage for the water and navigation, and be a great saving in materials and labour, as there will be fewer piers and centres, and the arches themselves will require less materials. And, one large single arch only is best, when it can be executed. For the fabric of a bridge, and the proper estimate of the expence, &c, there are generally necessary three plans, three sections, and an elevation. The three plans, are so many horizontal sections, viz, the first a plan of the foundation under the piers, with the particular circumstances attending it, whether of gratings, planks, piles, &c: the second, is the plan of the piers and arches, &c: the third, is the plan of the superstructure, with the paved road and banquet. The three sections, are vertical ones: the first of them a longitudinal section, from end to end, and through the middle of the breadth: the second, a transverse one, or across it, and through the summit of an arch: and the third also across, but taken on a pier. The elevation, is an orthographic projection of one side or face of the bridge, or its appearance as viewed at a great distance, showing the exterior aspect of the materials, and the manner in which they are worked and decorated.—Other observations are to be seen in the first section.

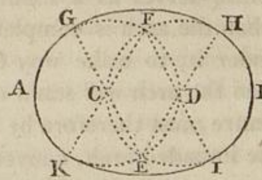
Butments, or Abutments, are the extremities of a bridge, by which it joins to, or abuts on, the land or sides of the river, &c. These must be made very secure, quite immovable, and more than barely sufficient to resist the drift of its adjacent arch. So that, if there are not rocks or very solid banks to raise them against, they must be well reinforced with proper walls or returns, &c. The thickness of them, which will be barely sufficient to resist the shoot of the arch, may be calculated as that of a pier by prop. xi.

When the foundation of a butment is raised against a sloping bank of rock, gravel, or good solid earth, it will produce a saving of materials and labour, to carry the work on by returns at different heights against it, like steps of stairs. And if the foundation, and all the courses, parallel to it, be laid, not horizontal, but rising backwards, so as to be perpendicular to the springing and pressure of the arch, it will be less liable to slide or be forced back by the push of the arch.

CAISSON, a kind of CHEST, or flat-bottomed boat, in which a pier is built, then sunk to the bed of the river, and the sides loosened and taken off from the bottom, by a contrivance for that purpose; the bottom of it being left under the pier as a foundation. It is evident therefore, that the bottoms of caissons must be made very strong, and fit for foundations of the piers. The caisson is kept afloat till the pier be built to about the height of low-water mark; and, for that purpose, its sides must either be made of more than that height at first, or else gradually raised to it as it sinks by the weight of the work, so as always to keep its top above water. And therefore the sides must be made very strong, and be kept asunder by cross timbers within, lest the great pressure of the ambient water should crush the sides in, and so not only endanger the work, but also drown the men who work within it. The caisson is made of the shape of the pier, but some feet wider on every side, to make room for the men to work: the whole of the sides are of two pieces, both joined to the bottom quite around, and to each other at the salient angles, so as to be disengaged from the bottom, and from each other, when the pier is raised to the desired height, and sunk. It is also convenient to have a small sluice made in the bottom, occasionally to open and shut, to sink the caisson and pier sometimes by, before it be finished, to try if it bottom level and rightly; for, by opening the sluice, the water will rush in and fill it to the height of the exterior water, and the weight of the work already built will sink it; then, by shutting the sluice, and pumping out the water, it will be made to float again, and the rest of the work may be completed: but it must not be sunk except when the sides are high enough to reach above the surface of the water, otherwise it cannot be raised and laid dry again. Mr. Labelye says, that the caissons in which he built some of the piers of Westminster bridge, contained above 150 load of fir timber, of 40 cubic feet each, and was of more tonnage, or capacity, than a 40 gun ship of war.

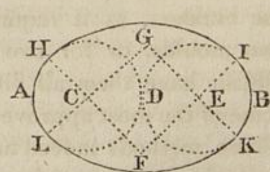
CENTRES, and CENTRING, or CENTERING, are the timber frames erected in the spaces of the arches, to turn them on, by building on them the voussoirs of the arches. As the centre serves as a foundation for the arch to be built on, when the arch is completed, that foundation is struck from under it, to make way for the water and navigation, and then the arch will stand of itself from its curved figure. A centre must therefore be constructed of the exact figure of the intended arch, convex as the arch is concave, to receive it on as a mould. If the form be circular, the curve is struck from a central point by a radius: if it be elliptical, it ought to be struck with a doubled cord, passing over two pins or nails fixed in the foci, as the mathematicians and gardeners describe their ellipses. Very often, in practice, an oval is employed, as made of three circular arcs. This very nearly resembles the true geometrical ellipsis, being formed of two equal arcs of small circles at the extremities, having between them a longer arch of a much larger circle, the ends of these arches being made to butt and join to each other, that they seem like the same curve only continued. As this mechanical oval will have nearly the same properties and effect as the true ellipsis, and can be more conveniently worked by the builders, as it requires the voussoirs to be cut only to two moulds, or for two centres, while those for the true ellipsis have them all different, we shall add in this place some of the most approved methods of describing these ovals. These methods indeed are, and must be, various, according as the length or span is required to be more or less, in proportion to the breadth or height. But in all of them, the centres of the large and small arcs must be so taken, that the right line passing through them, may also, when continued, pass through exactly the point where the ends of those arches butt and join together; for by this means they will have the same common tangent at that point, and consequently they will unite together, or run into each other, like parts of the same curve produced.

FIRST METHOD.—*When the Length and Breadth differ not very much.*



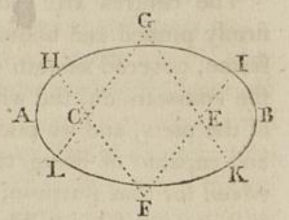
Divide the given length or span AB into three equal parts, at the points C and D. With one of those parts, CD, as a radius, and from the two centres C, D, describe two circles, intersecting each other in the two points E and F. Through these two points E, F, and the two centres C, D, draw four lines ECG, EDH, FDI, FCK, cutting the two circles in the four points G, H, I, K. Lastly, with one of these lines, as a radius, and from the two centres E, F, describe the two arches GH, KI, and they will complete the oval, forming a figure so much resembling a true ellipse, that the eye cannot perceive the difference between them. In this oval it is evident that the radius of the larger circular arch is just double of that of the smaller arches.

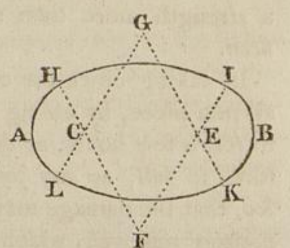
SECOND METHOD.—*For a Narrower Oval.*



Divide the length or span AB into four equal parts; then, with one of those parts as a radius, and from the three points of division, C, D, E, as centres, describe three circles. Find the uppermost and lowest points, F, G, of the middle circle; or through the middle point D draw a perpendicular to AB, which will give the points F, G, or construct the square CGEF, which will give the centres of the larger arch. Through these two points F, G, and the two C, E, draw four lines FH, FI, GK, GL; with any one of which as a radius, and the two centres F, G, describe the other two arcs HI, KL, to complete the oval; which does not rise so high as the former.

THIRD METHOD.





Other ovals may be made to the same length, or any other length, but rising still less in the crown, in any degree whatever, if, after having described the two smaller or end circles from the centres C and E, as in the second method, instead of forming the right angled triangles CGE, CFE, these be described with acute angles at F and G, by making the equal lines CF, CG, EF, EG, longer than before in any ratio at pleasure; these being then produced to the little circles at the four points H, I, K, L, from the centres F, G, describe the other two arches HI and KL, to complete the ovals, narrower and narrower at pleasure.

The little circles also at the ends, may have their radius taken smaller to any degree, or a less portion of the whole span; and indeed it is evident that its radius ought always to be less than the pitch or height of the arch.

There are other methods of making such ovals, but those above given are some of the best. The last method is general too, and will serve to accommodate an oval to any length and breadth whatever, at pleasure. Having thus described the half of such an oval to any span and pitch proposed, for any arch of a bridge, &c, the whole of the voussoirs may be cut by two mold boards only, viz, one for the voussoirs for the arch AH and IB, and the other for those in the arch HI.

But if the arch be of any other form, the several abscisses and ordinates ought to be calculated; then their corresponding lengths, transferred to the centring, will give so many points of the curve, and exactly by these points bending a bow of pliable matter, the curve may be drawn close by it.

The centres are constructed of beams, &c, of timber, firmly pinned and bound together, into one entire compact frame, covered smooth at top with planks or boards to place the voussoirs on, the whole supported by offsets in the sides of the piers, and by piles driven into the bed of the river, and capable of being raised and depressed by wedges, contrived for that purpose, and for taking them down when the arch is completed. They ought also to be constructed of a strength more than sufficient to bear the weight of the arch.

In taking down the centring; it is first let down a little, all in a piece, by easing some of the wedges; it is there let to rest a few hours, or days, to try if the arch make any efforts to fall, or any joints open, or stones crush or crack, &c, that the damage may be repaired before the centring is entirely removed, which is not to be done till the arch ceases to make any visible efforts.

In some bridges the centring makes a considerable part of the expence, and therefore all means of saving in this article ought to be closely attended to; such as making few arches, and as nearly alike or similar as possible, that the centring of one arch may serve for others, and at least that the same centre may be used for each pair of equal arches, on both sides of the middle.

CHEST, the same as CAISSON.

COFFERDAM, the same as BATTERDEAU.

DRIFT, SHOOT, or THRUST, of an arch, is the push or force which it exerts in the direction of the length of the bridge. This force arises from the perpendicular gravitation or weight of the stones of the arch, which, being kept from descending by the form of the arch and the resistance of the piers, exert their force in a lateral direction. This force is computed in prop. XI, where the thickness of the pier is determined which is necessary to resist it; and is the greater as the pitch is lower, *cæteris paribus*.

Elevation, the orthographic projection of the front of a bridge, on the vertical plane, parallel to its length. This is necessary to show the form and dimensions of the arches, and other parts, as to height and breadth, and therefore it has a plain scale annexed to it, to measure the parts by. It also shows the manner of working up and decorating the fronts of the bridge.

Extrados, the exterior curvature or line of an arch. In the propositions of the second section, it is the outer or upper line of the wall above the arch; but it often means only the upper or exterior curve of the voussoirs.

Foundations, the bottoms of the piers, &c, or the bases on which they are built. These bottoms are always to be made with projections, greater or less according to the spaces on which they are built. And according to the nature of the ground, the depth and velocity of water, &c, the foundations are laid, and the piers built, after different manners, either in caissons, in batterdeaux, or on stilts with sterlings, &c; for the particular methods of doing which, see each under its respective term.

The most obvious and simple method of laying the foundations, and raising the piers up to water-mark, is to turn the river out of its course above the place of the bridge, into a new channel, cut for it near the place where it makes an elbow or turn; then the piers are built on dry ground, and the water turned into its old course again, the new one being securely banked up. This is certainly the best method, when the new channel can be easily and conveniently made; but which however is very seldom the case.

Another method is, to lay only the space of each pier dry, till it be built, by surrounding it with piles and planks driven down into the bed of the river, so close together as to exclude the water from coming in; then the water is pumped out of the inclosed space, the pier built in it, and lastly the piles and planks drawn up. This is cofferdam work; but it evidently cannot be practised when the bottom is of a loose consistence, admitting the water to ooze and spring up through it.

When neither the whole nor part of the river can be easily laid dry, as above, other methods are to be used; such as, to build either in caissons or on stilts, both which methods are described under their proper words; or yet by another method, which hath, though seldom, been sometimes used, without laying the bottom dry, and which is thus: the pier is built upon strong rafts or gratings of timber, well bound together, and buoyed up on the surface of the water by strong cables, fixed to other floats or machines, till the pier is built; the whole is then gently let down to the bottom, which must be made level for the purpose. But of these methods, that of building in caissons is the best.

But before the pier can be built in any manner, the ground at the bottom must be well secured, and made quite good and safe, if it be not so naturally. The space must be bored into, to try the consistence of the ground; and if a good bottom of stone, or firm gravel, clay, &c, be met with, within a moderate depth below the bed of the river, the loose sand, &c, must be removed and digged out to it, and the foundation laid on the firm bottom, on a strong grating, or base of timber, made much broader every way than the pier, that there may be the greater base to press on, to prevent its being sunk. But if a solid bottom cannot be found at a convenient depth to dig to, the space must then be driven full of strong piles, the tops of which must be sawed off level, some feet below the bed of the water, the sand having been previously digged out for that purpose; and then the foundation, on a grating of timber, laid on their tops as before. Or, when the bottom is not good, if it be made level, and a strong grating of timber, two, three, or four times as large as the base of the pier, be made, it will form a good base to build on, its great size in a great measure, preventing it from sinking. In driving the piles, the method is, to begin at the middle, and proceed outwards, all the way to the borders or margin: the reason of which is, that if the outer piles were driven first, the earth of the inner space would be thereby so jammed together, as not to allow the inner piles to be driven at all. And besides the piles immediately under the piers, it is also very prudent to drive in a single, double, or triple row of them, around and close to the frame of the foundation, cutting them off a little above it, to secure it from slipping aside out of its place, and to bind the ground under the pier the firmer. For, as the safety of the whole bridge depends much on the foundations, too much care cannot be used to have the bottom made quite secure.

JETTEE, the border made around the stilts under a pier; being the same with STERLING.

IMPOST, is the part of the pier on which the feet of the arches stand, or from which they spring.

KEYSTONE, the middle voussoir, or the arch stone in the crown, or immediately over the centre of the arch. The length of the keystone, or thickness of the archivolt at top, is allowed to be about 1-15th or 1-16th of the span, by the best architects.

ORTHOGRAPHY, the elevation of a bridge, or front view, as seen at a great distance.

PARAPET, the breast wall made on the top of a bridge, to prevent passengers from falling over. In good bridges, to build the parapet only a little part of its height close or solid, and on that a balustrade to above a man's height, has an elegant and useful effect.

PIERS, are the walls built for the support of the arches, and from which they spring as their bases. These ought to be built of large blocks of stone, solid throughout, and cramped together with iron, or otherwise, which will make the whole like one solid stone. Their faces or ends, from the base up to high-water mark, ought to project sharp out with a salient angle, to divide the stream. Or perhaps the bottom of the pier should be built flat or square up to about half the height of low-water mark, to allow a lodgment against it for the sand or mud, to cover the foundation; lest, by being left bare, the water should in time undermine, and so ruin or injure it. The best form of the protection for dividing the stream, is the triangle; and the longer it is, or the more acute the salient angle, the better it will divide it, and the less will the force of the water be against the pier; but it may be sufficient to make that angle a right one, as it will make the work stronger, and in that case the perpendicular projection will be equal to half the breadth or thickness of the pier. In rivers on which large heavy craft navigate, and pass the arches, it may perhaps be better to make the ends semicircular; for though it does not divide the water so well as the triangle, it will both better turn off and bear the shock of the craft.

The thickness of the piers ought to be such, as will make them of weight or strength sufficient to support their interjacent arch, independent of any other arches. The thickness, in most cases of practice, may be made about $\frac{1}{8}$ of the span of the arch. And then, if the middle of the pier be run up to its full height, the centring may be struck, in order to be used in another arch, before the hanches are filled up. The whole theory of the piers may be seen in the third section. They ought to be made with a broad bottom on the foundation, and gradually diminished in thickness by offsets, up to low-water mark. The methods of laying their foundations, and building them up to the surface of the water, are given under the word FOUNDATION.

PILES, are timbers driven into the bed of the river for various purposes, and are either round, square, or flat like planks. They may be of any wood which will not rot under water, but elm, oak, and fir are mostly used, especially the latter, on account of its length, straightness, and cheapness. They are shod with a pointed iron at the bottom, the better to penetrate into the ground; and are bound with a strong iron band or ring at top, to prevent them from being split by the violent strokes of the ram by which they are driven down. It is said, that the stilts, or piles, under London-bridge, are of elm, which lasts a long time in the water.

Piles are either used to build the foundations on, or are driven about the pier as a border of defence, or to support the centres on; and in this case, when the centring is removed, they must either be drawn up, or sawed off very low under water; but it is perhaps better to saw them off, and leave them sticking in the bottom, lest the drawing of them out should loosen the ground about the foundation of the pier. Those to build on, are either such as are cut off by the bottom of the water, or rather a few feet within the bed of the river; or else such as are cut off at low-water mark, and then they are called stilts. Those to form borders of defence, are rows driven in close by the frame of a foundation, to keep it firm; or else they are to form a case or jettee about the stilts, to keep within it the stones that are thrown in to fill it up; in this case, the piles are grooved, driven at a small distance from each other, and plank piles let into the grooves between them, and driven down also, till the whole space is surrounded. Besides using this for stilts, it is also sometimes necessary to surround a stone pier with a sterling or jettee, and fill it up with stones to secure an injured pier from being still more damaged, and the whole bridge ruined. The piles to support the centres may also serve as a border of piling to secure the foundation, cutting them off low enough after the centre is removed.

Pile Driver, is an engine for driving down the piles. It consists of a large ram or square block of iron, sliding perpendicularly down between two guide posts; which being drawn up to the top of them, and there let fall from a great height, it comes down on the top of the pile with a violent blow. It is worked either by men or horses, and either with or without wheel work. That which was used at the building of Westminster-bridge, is perhaps one of the best kind.

Pitch, of an arch, is the perpendicular height from the spring, or impost, to the keystone.

Plan, of any part, as of the foundations, or piers, or superstructure, is the orthographic projection of it on a plane parallel to the horizon.

Push, of an arch, the same as drift, shoot, or thrust.

Salient Angle, of a pier, is the projection of the end against the stream, to divide it. The right-lined angle best divides the stream, and the more acute the better for that purpose; but the right angle is generally used, as making the best masonry. A semicircular end, though it does not divide the stream so well, is sometimes better in large navigable rivers, as it carries the craft the better off, or bears their shocks the better.

Shoot, of an arch, is the same as drift, thrust, &c.

Span, of an arch, is the extent or width at the bottom, or on the level at its springing.

Spandrels, or Spandrils, are the spaces about the flanks or haunches of the arch, above the curve or intrados.

Springers, are the first or lowest stones of an arch, being those at its feet, bearing immediately on the impost.

Sterlings, or Jettees, a kind of case, made of stilts, &c, about a pier, to secure it. It is particularly described under the next word Stilts.

Stilts, a set of piles driven into the space intended for the pier, whose tops being sawed level off about low-water mark, the pier is then raised on them. This method was formerly used, when the bottom of the river could not be laid dry; and these stilts were surrounded, at a few feet distance, by a row of piles and planks, &c, close to them like a coffer-dam, and called a sterling or jettee; after which, loose stones, &c, are thrown or poured down into the space, till it be filled up to the top, by that means forming a kind of pier of rubble or loose work, which is kept together by the sides of the sterlings: this is then paved level at the top, and the arches turned upon it. This method was formerly much used, most of the large old bridges in England being constructed in that way; such as London-bridge, Newcastle-bridge, Rochester-bridge, &c. But the inconveniencies attending it are so great, that it is now quite exploded and disused: for, because of the loose composition of the piers, they must be made very large or broad, otherwise the arch would push them over, and rush down as soon as the centre should be drawn: which great breadth of piers and sterlings so much contracts the passage of the water, as not only very much incommodes the navigation through the arch, from the fall and quick motion of the water, but from the same cause also the bridge itself is in much danger, especially in time of floods, when the quantity of water is too much for the passage. Add to this, that besides the danger there is of the pier bursting out the sterlings, they are also subject to much decay and damage by the rapidity of the water, and the craft passing through the arches.

THRUST, the same as drift, shoot, &c.

VOUSSOIRS, the stones which immediately form the arch, their under sides constituting the intrados or soffit. The middle one, or keystone, ought to be, in length, about $\frac{1}{15}$ or $\frac{1}{16}$ of the span, as has been observed; and the rest should increase in size all the way down to the impost; the more they increase the better, as they will the better bear the great weight which rests upon them, without being crushed, and also will bind the firmer together. Their joints should also be cut perpendicular to the curve of the intrados.

TRACT II.

QUERIES CONCERNING LONDON BRIDGE: WITH THE ANSWERS, BY GEORGE DANCE, ESQ.

AS an Appendix to the foregoing Tract, on the Principles of Bridges, a few smaller papers, on kindred subjects, are inserted in this and some of the Tracts immediately following. The present paper is one, among several of a curious nature, which I purchased at the sale of Mr. Robertson's books, in the year 1776, and appears to contain circumstances of too much importance to be kept private. It seems to have ori-