DE

SVPERFICIEBVS

SECVNDI ORDINIS

AVCTORE

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PARTICVLA PRIMA.

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TYPIS DIETERICHIANIS

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PIETATIS

MONIMENTVM EXIGVVM.

L. B. S.

Prima hac particula expositionis analyticae superficierum secundi ordinis elementa disciplinae de formis ternariis quadraticis, quam ill. *Gauss* exhibuit in Disquisitionibus Arithmeticis, quatenus quidem hoc loco emolumento esse visum est, ad contemplationes geometrici argumenti applicare adortus sum. Equidem enim censebam, nexum tum inter aequationem ac superficiem quum inter ipsas aequationes vel inter superficies ratione ista ad maiorem euehi perspicuitatem. Quin etiam introductis superficiebus adiunctis singularis quandoque comprobatur affinitas inter superficies vulgares atque irrepraesentabiles, illustranda suo loco in alia particula.

Aequatio superficiei alii cuidam adiunctae initio saltem ita sistenda mihi visa est, vt centra coincidant; postmodum vero, quum de superficiebus ipsis nec de aequationibus tantum agetur, vocabuli vis aliquantum amplianda erit neque vlli superficiei nomen adiunctae denegandum, cuius aequatio sub aequatione illa proprie adiuncta, si transformationem primam vtrimque peregeris, contenta sit implicatione normae 1. Relatiuus nimirum situs inepte stabiliret essentiale inter superficies vinculum.

Denominationum variarum, quae inde ab Euleri tempore pro singulis speciebus superficierum secundi ordinis in vsum venerunt, nonnullas ab auctoribus gallicis vsurpatas, ab aliis versione fere absona libris germanicis insertas cum leuioribus e principio analytico repetendis commutare forsan haud inutile foret. Etenim in omnibus cuiuscunque superficiei secundi ordinis punctis mensura curuaturae (conf. *Gauss* Disquisitiones generales circa superficies curuas) aut est 0 aut eodem signo praedita, vti opportuno loco probabitur. Itaque in casu posteriori superficiem appellare licebit aut conflexam aut difflexam, prout est aut concauo-concaua (siue, quod idem est, conuexo-conuexa) aut conuexo-concaua (sive concauo-conuexa). Quo pacto igitur ambarum hyperboloidum alteram quam vocant "à deux nappes", conflexam, alteram alias dicta "à une nappe" difflexam, eodemque modo paraboloideum alteram, quam vulgo ellipticam dicunt, conflexam, alteram, quae hyperbolica vel etiam interdum hyperbolois parabolica audire solet, paraboloidem difflexam vocare maluerim. In cylindris et cono, superficiebus in planum explicabilibus, mensura curuaturae vbique est 0, atque ibi denominationes istae adhiberi nequeunt. Cylindros igitur cognominibus solitis (ellipticum, hyperbolicum, parabolicum) distinguere proderit. At conus eodem iure, quo ellipticus, sicuti quidam voluere, etiam hyperbolicus vel parabolicus audiret; reapse nulla indiget appositione, nisi "secundi ordinis."

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Ioannes Benedictus Listing.

DE SVPERFICIEBVS SECVNDI ORDINIS.

1.

Aequiparanda cifrae quauis functione data trium quantitatum indeterminatarum x, y, z, in qua determinatae, quas insuper continet, sunt quantitates reales, exoritur aequatio, cui tanquam conditioni valores tribus indeterminatis impertiendi debent esse subacti. Quantitatibus indeterminatis x, y, z infinities aliae aliaeque terniones (vti dicere licitum sit) valorum qualiumcunque, realium seu imaginariorum conuenient aequationi institutae satisfacientes. Omnes hae terniones, re mere analytice considerata, in tribus complures dirimi poterunt combinationibus diuersis valorum realium, imaginariorum purorum et imaginariorum mixtorum *) superstruendas, quarum numerus, si inter indeterminatas permutationes arbitrariae admittuntur, $= 10$, sin fixus inter eas stabilitur ordo, $= 27$ erit.

2.

Quem in modum quamquam vastus analysi pateat inuestigationum theoreticarum campus, tamen methodus vsurpata in geometria analytica (praesertim in ea, quae solet amplecti tres spatii dimensiones) ad diligentiorem perquisitionem vnius

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*) vide *Gauss* Theoria residuorum biquadraticorum art. 31.

tantum restringitur omnium quos commemorauimus casuum, reliquos cunctos vtpote omni indigentes significatione geometrica in vnicum illi contrarium colligens ac supersedens accuratiori eorum explorationi. Simulatque enim tres valores reales h, k, l indeterminatis x, y, z resp. assignati denotant coordinatas puncti cuiusdam, sistentes positionem eius respectu systematis trium axium sub angulis determinatis in vno puncto sese decussantium, hae ipsae quantitates indeterminatae x, y, z significabunt eiusdemmodi coordinatas puncti alicuius positione indeterminati, quod tum tantum vspiam exstabit in spatio, quum ternio ipsis x, y, z tummaxime tributa tres continet valores reales, ceteroquin vero nusquam erit situm. Exempla vndique praesto sunt. Ternionum $2, 1, 2$; $2i, -i, 0$; $0, i, 1-i$; $-4, \frac{1}{8}-i, 2+\frac{1}{2}i$; $1-2i, 1-i, 1+2i$ (denotante i vnitatem imaginariam positiuam $\sqrt{-1}$) aequationi $x+2y-zz=0$ satisfacientium, quarum valor primus ipsi x, secundus ipsi y, tertius ipsi z impertiendus, sola prima positionem puncti cuiusdam determinat, quod reapse adest in spatio, ceterarum autem nulla puncto cuicunque existentiam vindicare — nedum positionem definire valet. Aequationi $\frac{1}{3}xx+\frac{2}{3}yy+\frac{9}{2}zz+1=0$ per ternionem trium valorum realium nequaquam satisfieri potest; nullibi itaque spatii datur punctum, cuius coordinatae sub ista aequatione comprehenderentur.

3.

Indita quantitatibus indeterminatis x, y, z variabilium indole, per datam inter x, y, z aequationem secundum legem continuitatis, certos saltem intra limites saluaque conditione art. praec. exhibita, determinari satis constat positionem complexus punctorum superficiem quandam constituentis. Dici solet, attinere superficiem ad aequationem vel per eam repraesentari. Quam secundam si seruare lubitum sit locutionem, haec repraesentatio in geometria analytica obuia ab ea, de qua in quibusdam arithmeticae sublimioris partibus agitur, probe distinguenda est. Discrimen vel maius est eo, quod intercedit inter aequationem atque functionem.

4.

Quoniam superficies secundi ordinis i. e. repraesentatae per aequationes quadraticas, de quibus his pagellis speciatim agere propositum est, ad algebraicas referendae sint, a scopo nostro prorsus alienum foret, transcendentes earumque expressiones analyticas silentio non praeterire. Contra nil vetiturum censuimus, quominus circa functiones algebraicas, in indaganda natura superficierum algebraicarum peculiare sibi appropriantes momentum, nonnulla eaque generaliora praemittantur. Perfacile est rem aliquantulo etiam maiore generalitate amplexu, quam esset necessarium, dum geometriae tantum finibus consuli deberet. Breuitatis gratia functiones algebraicae integrae rationales, ad quas hocce loco quaestionem oportet restringere *), in posterum plerumque simpliciter functiones audient.

Sit Φ functio completa (per quam forma generalis omnes functiones eiusdem gradus totidemque variabilium complectens intelligatur) gradus m inter n quantitates variabiles x, y, z, u etc.; m et n vel cifram vel numerum positiuum integrum quemuis designare possunt. Functionem Φ tam simplicem redditam esse supponimus, vt non contineat terminos aequales siue tantummodo quoad coëfficientes inaequales. Manifesto Φ aggregatum erit algebraicum numeri finiti terminorum formae $Kx^{\mathfrak{a}} y^{\mathfrak{b}} z^{\mathfrak{c}} u^{\mathfrak{d}} \ldots$, vbi exponentes $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ etc. vel numeros positiuos integros vel 0 denotant, summa autem eorum $\mathfrak{m}$ ipsum m superare nequit. Colligendo eos terminos, in quibus $\mathfrak{m}$ eodem praeditus est valore, denotandoque per $\varphi^{(\mathfrak{m})}$ eorum aggregatum, erit $\varphi^{(\mathfrak{m})}$ functio homogenea gradus $\mathfrak{m}$ inter n variabiles x, y, z, u etc. Vnde sequitur

$$\Phi = \varphi^{(0)} + \varphi^{(1)} + \varphi^{(2)} + \cdots + \varphi^{(m-1)} + \varphi^{(m)},$$

siue functio algebraica integra rationalis completa gradus m inter n quantitates variabiles est aggregatum $m+1$ functionum algebraicarum integrarum rationalium homogenearum omnium graduum a 0 vsque ad m.

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*) Quippe per annihilationem e functionibus aequationes conduntur, etiam saltem functionis fractae casum praesenti, quem respicimus, inuolui patet. Functiones autem irrationales prorsus diuersam diiudicationem requirunt.

Quodsi quis desideret, vt numeri terminorum functionum Φ, $\varphi^{(0)}$, $\varphi^{(1)}$ etc. innotescantur, facile inuenietur

functionem $\varphi^{(0)}$ habere 1 terminum (constantem),

$\varphi^{(1)}$ n terminos,

$\varphi^{(2)}$ $\frac{n(n+1)}{1\cdot 2}$,

etc. etc.

$\varphi^{(m-1)}$ $\frac{n(n+1)\ldots(n+m-2)}{1\cdot 2\cdot 3\ldots(m-1)}$,

$\varphi^{(m)}$ $\frac{n(n+1)\ldots(n+m-1)}{1\cdot 2\cdot 3\ldots m}$,

itaque summam Φ $\frac{(n+1)(n+2)\ldots(n+m)}{1\cdot 2\cdot 3\ldots m}$.

Functio homogenea e. g. quattuor variabilium x, y, z, u secundi gradus decem habet terminos, est enim haec $axx+byy+czz+duu+exy+fxz+gxy+hyz+kyu+lzu$. Completa quindecim continet terminos, nempe praeter allatos etiam hos $mx+ny+pz+qu+r$. Functio homogenea secundi gradus trium variabilium e sex, completa e decem constat terminis. Homogenea decimi gradus decem variabilium contineretur $\frac{10.11.12.13.14.15.16.17.18.19}{1.2.3.4.5.6.7.8.9.10.}$ siue 92378 terminos, completa autem duplam multitudinem.

Subleuandae perspicuitatis gratia nonnihil proderit, pro quibusdam functionum algebraicarum generibus in geometria curuarum et superficierum prae aliis frequenter obuiis quasdam adoptare denotationes abbreuiantes. Et homogeneae quidem functiones primi et secundi gradus duarum vel trium variabilium, nisi potissimum ad quantitates variabiles respicitur, per expingendos solos coëfficientes exhiberi debent. Hoc nimirum pacto etiam functiones, quas completas appellauimus, vti aggregata homogenearum satis simplices exsistent. Functio igitur homogenea *binaria* $Ax+A'y$ per

$$(A,\ A')$$

designetur, homogenea *ternaria* (trium variabilium) linearis $Ax+A'y+A''z$ per

$$(A,\ A',\ A''),$$

homogenea binaria secundi gradus siue quadratica $Axx + A'yy + 2Bxy$ *) per

$$\binom{A,\ A'}{B};$$

functio denique homogenea ternaria quadratica $Axx + A'yy + A''zz + 2Byz + 2B'zx + 2B''xy$ per

$$\begin{pmatrix} A, & A', & A'' \\ B, & B', & B'' \end{pmatrix}.$$

Inter quantitates variabiles, quarum in designationibus istis desunt literulae, ordo fixus conseruari debet, ita vt x pro prima, y pro secunda, z pro tertia haberi perseueret. Quo pacto e. g. $\binom{a,\ 1}{0}$ signum erit functionis binariae $axx + yy$, at $\binom{1,\ a}{0}$ functionis $xx + ayy$, similique modo $\begin{pmatrix} 1, & 1, & 1 \\ \frac{1}{2}, & 0, & 0 \end{pmatrix}$ functionis $xx + yy + zz + yz$, $\begin{pmatrix} 1, & 1, & 1 \\ 0, & \frac{1}{2}, & 0 \end{pmatrix}$ vero et $\begin{pmatrix} 1, & 1, & 1 \\ 0, & 0, & \frac{1}{2} \end{pmatrix}$ signa resp. functionum $xx + yy + zz + zx$ et $xx + yy + zz + xy$.

5.

Posito in art. anteced. $n = 3$, fiat $\varphi^{(0)} = f^{(0)}$, $\varphi^{(1)} = f^{(1)}$, $\varphi^{(2)} = f^{(2)}$ etc. positoque insuper $m = 2$, fiat $\Phi = W$. Erit itaque W functio algebraica rationalis integra completa secundi gradus trium variabilium, adeoque prodibit aequatio repraesentans superficies secundi ordinis maximaque gaudens generalitate

$$W = 0,$$

vbi $W = f^{(2)} + f^{(1)} + f^{(0)}$. Quodsi faciamus

$$f^{(2)} = Axx + A'yy + A''zz + 2Byz + 2B'zx + 2B''xy,$$
$$f^{(1)} = 2Cx + 2C'y + 2C''z,$$
$$f^{(0)} = -K,$$

*) Commodum in functionibus quadraticis binariis atque ternariis dimidia tantum coëfficientium terminorum eorum, qui producta continent e variabilibus, in denotationes istas recipiendi infra sponte elucebit.

denotantibus A, A', A'', B, B', B'', C, C', C'', K quantitates determinatas quascunque reales, aequatio generalis superficierum secundi ordinis haec erit

$$(1) \qquad Axx + A'yy + A''zz + 2Byz + 2B'zx + 2B''xy + 2Cx + 2C'y + 2C''z = K,$$

quam adiumento designationum art. anteced. prolatarum ita exhibemus

$$(2) \qquad \binom{A,\ A',\ A''}{B,\ B',\ B''} + 2(C,\ C',\ C'') = K.$$

6.

Systema coordinatarum rectilinearium, ad quod determinanda auxilio aequationis (2) superficies debet referri, maxima quoque praeditum est generalitate. Tres axes in vno puncto se intersecantes angulos determinatos quoscunque efficiunt. Quo in casu generali systema coordinatarum *obliquangulare* seu *obliquum*, sin anguli, quibus axes inter se coniunguntur, omnes sunt recti, *rectangulare* seu *rectum* appellari constat, ita vt hoc sub illo quasi species sub genere contineatur. Per operationem, quae transformatio coordinatarum dici solet, aequatio (2) in alias commutari potest, aliis superstructas systematibus, superficie ipsa tamen penitus manente inuariata. Priusquam analogias inuestigamus, quarum ope huiusmodi transformationes aequationis (2) impetrandae sunt, nonnullas denotationes aliorsim plus minusue adoptatas hic praemittamus, quibus vtentes aliquantum quandoque verborum prolixitatis valebimus euitare. Etenim axem (x) eum vocemus axis coordinatarum x ramum, qui e communi axium intersectione siue ex puncto initiali in eam plagam versus tendit, quam versus coordinatae x crescunt. Planum (xy) sit planum fundamentale axes (x) et (y) continens. Angulus inter axes (x) et (y) interceptum per (x, y), angulus inter axem (x) et planum (yz) per (x, yz) denotetur, eodemque modo angulus, quo plana (xy), (xz) in se inuicem inclinata sunt, per (xy, xz) potest signari. Systema axium (x), (y), (z) simpliciter systema

(xyz), punctum denique, cuius coordinatae sunt x, y, z, punctum x, y, z audiat. Atque similes similibus notis insint notiones.

Iam ad eruendas formulas, quibus, quam repraesentat aequatio (2), superficies ad aliud quoduis systema coordinatarum obliquum transgeratur, primo cum respiciamus casum, quo noui systematis coordinatarum primariique puncta initialia in vnum coïncidant, vnde alter generalior conditione ista exsors facile promanabit. Sit itaque vtrique systemati, primo (xyz) ac secundo $(\xi\eta\zeta)$ commune punctum initiale, concipianturque posita plana tria, plano (yz) parallela, primum per punctum $\xi, 0, 0$, alterum per punctum $\xi, \eta, 0$, tertium per punctum ξ, η, ζ, quod simul per punctum $x, 0, 0$ transibit. Tunc est

$$\begin{aligned} \text{distantia plani primi a plano } (yz) &= \xi \cdot \sin(\xi, yz), \\ \text{plani secundi a primo} &= \eta \cdot \sin(\eta, yz), \\ \text{plani tertii a secundo} &= \zeta \cdot \sin(\zeta, yz), \\ \text{denique distantia plani tertii a plano } (yz) &= x \cdot \sin(x, yz). \end{aligned}$$

Quoniam autem vltima distantia est aggregatum algebraicum trium priorum, prodibunt, ratiocinio etiam pro planis (zx), (xy) reiterato, formulae sequentes

$$x = \frac{\sin(\xi, yz)}{\sin(x, yz)}\xi + \frac{\sin(\eta, yz)}{\sin(x, yz)}\eta + \frac{\sin(\zeta, yz)}{\sin(x, yz)}\zeta,$$

$$y = \frac{\sin(\xi, zx)}{\sin(y, zx)}\xi + \frac{\sin(\eta, zx)}{\sin(y, zx)}\eta + \frac{\sin(\zeta, zx)}{\sin(y, zx)}\zeta,$$

$$z = \frac{\sin(\xi, xy)}{\sin(z, xy)}\xi + \frac{\sin(\eta, xy)}{\sin(z, xy)}\eta + \frac{\sin(\zeta, xy)}{\sin(z, xy)}\zeta.$$

Pro casu generaliori, quo suum vtrique systemati attribuendum est punctum initiale, sint axes (x'), (y'), (z') noui eiusque tertii systematis coordinatarum axibus (ξ), (η), (ζ) resp. paralleli atque vna cum iis ad eandem plagam versus tendentes. Quo pacto, si d, d', d'', quae designent quantitates determinatas quasuis reales, sunt coordinatae initii systematis $(x'y'z')$ relati ad systema $(\xi\eta\zeta)$, in expressionibus modo inuentis substituere sufficit $x' + d$ pro ξ, $y' + d'$ pro η, $z' + d''$ pro ζ.

Formulas igitur, quibus ducibus a systemate obliquo (xyz) ad aliud quodcunque obliquum $(x'y'z')$ queamus transgredi, hancce speciem induere satis elucebit

$$(3)\qquad \begin{cases} x = \alpha x' + \beta y' + \gamma z' + \delta\,, \\ y = \alpha' x' + \beta' y' + \gamma' z' + \delta'\,, \\ z = \alpha'' x' + \beta'' y' + \gamma'' z' + \delta''\,, \end{cases}$$

vbi $\alpha, \beta, \gamma, \alpha', \beta', \gamma', \alpha'', \beta'', \gamma'', \delta, \delta', \delta''$ sunt quantitates determinatae reales.

Si ipsam substitutionem valorum (3) pro coordinatis x, y, z in aequatione (1) persequi velis, inuenies, quum posueris

$$(4)\qquad \begin{cases} L\ = A\alpha\alpha + A'\alpha'\alpha' + A''\alpha''\alpha'' + 2B\alpha'\alpha'' + 2B'\alpha''\alpha + 2B''\alpha\alpha'\,, \\ L'\ = A\beta\beta + A'\beta'\beta' + A''\beta''\beta'' + 2B\beta'\beta'' + 2B'\beta''\beta + 2B''\beta\beta'\,, \\ L''= A\gamma\gamma + A'\gamma'\gamma' + A''\gamma''\gamma'' + 2B\gamma'\gamma'' + 2B'\gamma''\gamma + 2B''\gamma\gamma'\,, \\ M\ = A\beta\gamma + A'\beta'\gamma' + A''\beta''\gamma'' + B(\beta'\gamma'' + \gamma'\beta'') + B'(\beta''\gamma + \gamma''\beta) + B''(\beta\gamma' + \gamma\beta')\,, \\ M'\ = A\gamma\alpha + A'\gamma'\alpha' + A''\gamma''\alpha'' + B(\gamma'\alpha'' + \alpha'\gamma'') + B'(\gamma''\alpha + \alpha''\gamma) + B''(\gamma\alpha' + \alpha\gamma')\,, \\ M''= A\alpha\beta + A'\alpha'\beta' + A''\alpha''\beta'' + B(\alpha'\beta'' + \beta'\alpha'') + B'(\alpha''\beta + \beta''\alpha) + B''(\alpha\beta' + \beta\alpha')\,, \\ N\ = A\alpha\delta + A'\alpha'\delta' + A''\alpha''\delta'' + B(\alpha'\delta'' + \delta'\alpha'') + B'(\alpha''\delta + \delta''\alpha) + B''(\alpha\delta' + \delta\alpha') + C\alpha + C'\alpha' + C''\alpha''\,, \\ N'\ = A\beta\delta + A'\beta'\delta' + A''\beta''\delta'' + B(\beta'\delta'' + \delta'\beta'') + B'(\beta''\delta + \delta''\beta) + B''(\beta\delta' + \delta\beta') + C\beta + C'\beta' + C''\beta''\,, \\ N''= A\gamma\delta + A'\gamma'\delta' + A''\gamma''\delta'' + B(\gamma'\delta'' + \gamma'\delta'') + B'(\gamma''\delta + \delta''\gamma) + B''(\gamma\delta' + \delta\gamma') + C\gamma + C'\gamma' + C''\gamma''\,, \\ -Q = A\delta\delta + A'\delta'\delta' + A''\delta''\delta'' + 2B\delta'\delta'' + 2B'\delta''\delta + 2B''\delta\delta' + 2C\delta + 2C'\delta' + 2C''\delta'' - K\,, \end{cases}$$

aequationem (1) transmutatum iri in sequentem

$$(5)\quad Lx'x' + L'y'y' + L''z'z' + 2My'z' + 2M'z'x' + 2M''x'y' + 2Nx' + 2N'y' + 2N''z' = Q,$$

siue derelictis variabilibus

$$(6)\qquad \begin{pmatrix} L, & L', & L'' \\ M, & M', & M'' \end{pmatrix} + 2(N, N', N'') = Q,$$

cuius coëfficientes $L, L', L'', M, M', M'', N, N', N'', Q$, proinde atque aequationis (2), manifesto sunt quantitates determinatae reales. Vnde concludimus aequationes, quibus superficies quaedam secundi ordinis repraesentatur, diuersis quantumuis coordinatarum systematibus (xyz), $(x'y'z')$ superstructas sub forma generali (2) vsque contineri.

7.

Ex expressionibus generalibus (3) transformationi aequationis (2) per introducendum nouum coordinatarum systema inseruientibus speciales quascunque reuocandos esse per se claret. Ad eum igitur, quem vt respiciamus nostra proxime intererit, repetendum casum e formulis illis, iam paullo antea inter systemata $(\xi\eta\zeta)$ et $(x'y'z')$ obiter exortum, supponamus vtriusque, primarii nouique, systematis axes (x), (y), (z) atque (x'), (y'), (z') resp. parallelos esse et quoad regiones, quorsum tendunt, similiter iacere. Coordinatae puncti initii coordinatarum x', y', z' relati ad systema primarium (xyz) sint, veluti supra, d, d', d'', et posito generaliter

$$(7)\quad \begin{cases} \sin(x',yz) = \lambda\,, & \sin(y',yz) = \mu\,, & \sin(z',yz) = \nu\,, \\ \sin(x',zx) = \lambda'\,, & \sin(y',zx) = \mu'\,, & \sin(z',zx) = \nu'\,, \\ \sin(x',xy) = \lambda''\,, & \sin(y',xy) = \mu''\,, & \sin(z',xy) = \nu''\,, \\ \sin(x\,,yz) = \lambda^0, & \sin(y\,,zx) = \mu^0, & \sin(z\,,xy) = \nu^0, \end{cases}$$

formulae (3) praebebunt valores

$$(8)\quad \begin{cases} \alpha = \frac{\lambda}{\lambda^0}, & \beta = \frac{\mu}{\lambda^0}, & \gamma = \frac{\nu}{\lambda^0}, & \delta = \alpha\, d + \beta\, d' + \gamma\, d'', \\ \alpha' = \frac{\lambda'}{\mu^0}, & \beta' = \frac{\mu'}{\mu^0}, & \gamma' = \frac{\nu'}{\mu^0}, & \delta' = \alpha'\, d + \beta'\, d' + \gamma'\, d'', \\ \alpha'' = \frac{\lambda''}{\nu^0}, & \beta'' = \frac{\mu''}{\nu^0}, & \gamma'' = \frac{\nu''}{\nu^0}, & \delta'' = \alpha'' d + \beta'' d' + \gamma'' d'', \end{cases}$$

Quandoquidem vero nostro casu in (7) statuendum est $\lambda = \lambda^0$, $\mu' = \mu^0$, $\nu'' = \nu^0$, $\lambda' = \lambda'' = \mu'' = \mu = \nu = \nu' = 0$, habebitur ex (8)

$$(9)\quad \begin{cases} \alpha = 1, & \beta = 0, & \gamma = 0, & \delta = d\,, \\ \alpha' = 0, & \beta' = 1, & \gamma' = 0, & \delta' = d'\,, \\ \alpha'' = 0, & \beta'' = 0, & \gamma'' = 1, & \delta'' = d'', \end{cases}$$

ideoque reuera pro suppositione praesenti formulae (3) transeunt in has

$$(10)\qquad \begin{cases} x = x' + d\ , \\ y = y' + d'\ , \\ z = z' + d''. \end{cases}$$

Ad perficiendam ipsam valorum istorum in aequatione (1) substitutionem, per quam aequatio (1) nunc transire debet in aliam (5), adiumento valorum (9) ex expressionibus (4) sequitur

$$L = A,\ \ L' = A',\ \ L'' = A'',\ \ M = B,\ \ M' = B',\ \ M'' = B'',$$

$$(11)\qquad \begin{cases} N = A\,d + B''d' + B'\,d'' + C\ , \\ N' = B''d + A'\,d' + B\,\,d'' + C', \\ N'' = B'\,d + B\,\,d' + A''d'' + C'', \end{cases}$$

$$-Q = Add + A'd'd' + A''d''d'' + 2Bd'd'' + 2B'd''d + 2B''dd' + 2Cd + 2C'd' + 2C''d'' - K.$$

8.

Aequationes quummaxime erutae quaestioni debent inseruire, quatenus fieri possit, vt introducendo valores (10), in quibus ad hunc finem differentiae d, d', d'' inter primarii nouique systematis coordinatas tamquam indeterminatae considerandae sunt, ex aequatione generali superficierum secundi ordinis $f^{(2)} + f^{(1)} + f^{(0)} = 0$ pars linearis $f^{(1)}$ exterminetur, siue vt transeat aequatio proposita in talem $\binom{L,\ L',\ L''}{M,\ M',\ M''} = Q$. Namque si tentaminis gratia supponimus $N = N' = N'' = 0$, ex (11) ad determinandas ipsas d, d', d'' habemus aequationes

$$0 = A\,d + B''d' + B'\,d'' + C\ ,$$
$$0 = B''d + A'\,d' + B\,d'' + C',$$
$$0 = B'\,d + B\,d' + A''d'' + C'',$$

e quibus per eliminationem inuenietur

$$(12)\qquad \begin{cases} d = -\dfrac{(BB - A'A'')C + (A''B'' - BB')C' + (A'B' - B''B)C''}{ABB + A'B'B' + A''B''B'' - AA'A'' - 2BB'B''}, \\ d' = -\dfrac{(A''B'' - BB')C + (B'B' - A''A)C' + (AB - B'B'')C''}{ABB + A'B'B' + A''B''B'' - AA'A'' - 2BB'B''}; \\ d'' = -\dfrac{(A'B' - B''B)C + (AB - B'B'')C' + (B''B'' - AA')C''}{ABB + A'B'B' + A''B''B'' - AA'A'' - 2BB'B''}, \end{cases}$$

Hosce valores coordinatarum d, d', d'' resp. per g, g', g'' atque quantitatem $ABB + A'B'B' + A''B''B'' - AA'A'' - 2BB'B''$ per D designabimus.

Pendebit igitur a quantitate D, vtrum d, d', d'' ad determinationem puncti, a quo noui systematis $(x'y'z')$ coordinatae debent inchoare, idoneae sint, necne. Valores enim isti e postulatione proposita emanentes — semper quidem reales — erunt aut determinati aut non, prout D a cifra discrepat aut nihilo aequiualet. Ac proinde a valore ipsius D pendebit indoles quantitatis Q in aequatione (6). Tunc enim tantum, quum D cifrae est inaequalis, Q euadet determinata. Vnde sequitur, simulatque in aequatione (2) superficiei cuiusdam secundi ordinis quantitas D a cifra est diuersa, aequationem (2) per substitutionem formularum (10) transigi posse in hanc

$$\begin{pmatrix} L, & L', & L'' \\ M, & M', & M'' \end{pmatrix} = Q,$$

in qua L, L', L'', M, M', M'', Q sunt quantitates determinatae reales; simulatque vero D est cifrae aequalis, aequationem (2) tali modo transformari non posse.

Quam transformationem solummodo in casu D non $= 0$ applicabilem *primam* dicemus. Quantitati D autem siue $ABB + A'B'B' + A''B''B'' - AA'A''$ $- 2BB'B''$ quum in doctrina de „formis" quadraticis ternariis *) tum in theoria superficierum secundi ordinis momentum insigne sibi asserenti nomen *determinantis* illic receptum etiam hic conseruare licebit.

---

*) vide *Gauss* Disquisitionum arithmeticarum art. 267.

9.

Itaque transformatio prima in eo consistit, quod aequatio superficiei cuiusdam secundi ordinis inter variabiles x, y, z

$$(2) \qquad \begin{pmatrix} A, & A', & A'' \\ B, & B', & B'' \end{pmatrix} + 2(C, C', C'') = K,$$

cuius determinans $ABB + A'B'B' + A''B''B'' - AA'A'' - 2BB'B'' = D$ a cifra diuersus est, statuto

$$(13) \qquad \begin{cases} g = -\dfrac{(BB - A'A'')C + (A''B'' - BB')C' + (A'B' - B''B)C''}{D}, \\ g' = -\dfrac{(A''B'' - BB')C + (B'B' - A''A)C' + (AB - B'B'')C''}{D}, \\ g'' = -\dfrac{(A'B' - B''B)C + (AB - B'B'')C' + (B''B'' - AA')C''}{D}, \end{cases}$$

per substitutionem formularum

$$(14) \qquad \begin{cases} x = x' + g, \\ y = y' + g', \\ z = z' + g'' \end{cases}$$

transit in hanc

$$(15) \qquad \begin{pmatrix} A, & A', & A'' \\ B, & B', & B'' \end{pmatrix} = K',$$

variabilium x', y', z', determinantis eiusdem D, et in qua est

$$(16) \quad K' = K - Agg - A'g'g' - A''g''g'' - 2Bg'g'' - 2B'g''g - 2B''gg' - 2Cg - 2C'g' - C''g''.$$

Praeterea tenendum est, amborum coordinatarum systematum (xyz), $(x'y'z')$ axes respondentes esse tum parallelos tum (vti expositum est art. 6) similiter iacentes, determinarique systematis $(x'y'z')$ initium relatum ad ipsum (xyz) per coordinatas $x = g$, $y = g'$, $z = g''$.

Vt quosdam addamus casus speciales, proposita sit primo aequatio

$$2Byz + 2B'zx + 2B''xy + 2Cx + 2C'y + 2C''z = K$$

determinantis D a cifra diuersi. Siquidem habetur $A = A = A'' = 0$, erit $D = -2BB'B''$ euadentque

$$g = -\frac{BC - B'C' - B''C''}{2B'B''},$$

$$g' = -\frac{B'C' - B''C'' - BC}{2B''B},$$

$$g'' = -\frac{B''C'' - BC - B'C'}{2BB'}.$$

Secundo loco transformanda sit aequatio

$$(17) \qquad Axx + A'yy + A''zz + 2Cx + 2C'y + 2C''z = K$$

determinantis D a cifra diuersi. Exstant $B = B' = B'' = 0$ et $D = -AA'A''$, ideoque

$$(18) \qquad g = -\frac{C}{A}, \quad g' = -\frac{C'}{A'}, \quad g'' = -\frac{C''}{A''}.$$

Transibit igitur aequatio data per transformationem primam, adhibitis formulis

$$(19) \qquad \begin{cases} x = x' - \dfrac{C}{A}, \\ y = y' - \dfrac{C'}{A'}, \\ z = z' - \dfrac{C''}{A''}, \end{cases}$$

in hanc

$$(20) \qquad Ax'x' + A'y'y' + A''z'z' = K + \frac{CC}{A} + \frac{C'C'}{A'} + \frac{C''C''}{A''}.$$

Proficiscendo ab aequatione (15) determinantis non $= 0$ referente superficiem datam ad systema $(x'y'z')$ haec alia

$$\begin{pmatrix} A, & A', & A'' \\ B, & B', & B'' \end{pmatrix} + 2(H, H', H'') = J$$

eiusdem determinatis D, eandem superficiem repraesentans referensque eam ad systema $(x''y''z'')$, cuius axes cum respondentibus illius $(x'y'z')$ sunt paralleli ac similiter iacentes, poterit deriuari per substitutionem formularum $x' = x'' - h$, $y' = y'' - h'$, $z' = z'' - h''$, quo pacto erit

$$\begin{aligned} H &= -A\,h - B''h' - B'\,h'', \\ H' &= -B''h - A'\,h' - B\,h'', \\ H'' &= -B'\,h - B\,h' - A''h'', \\ J &= K' - Ahh - A'h'h' - A''h''h'' - 2Bh'h'' - 2B'h''h - 2B''hh'. \end{aligned}$$

Haec noua aequatio per transformationem primam auxilio formularum $x'' = x' + h$, $y'' = y' + h'$, $z'' = z' + h''$ instituendam necessario redire debet in pristinam (15). Quippe hac in deductione coordinatae $-h$, $-h'$, $-h''$, quibus positio puncti initialis systematis $(x''y''z'')$ definitur, penitus sunt arbitrariae, patet, infinite multas aequationes determinantium aequalium dari superficiem eandem secundi ordinis repraesentantes eamque ad diuersa systemata axium resp. parallelorum similiterque iacentium referentes, quae omnes, dumne determinans cunctis communis cifrae sit aequalis, per transformationem primam redeant ad certam aequationem vnicam.

10.

Data sit aequatio superficiei cuiusdam secundi ordinis inter coordinatas systematis (xyz)

$$\binom{A,\ A',\ A''}{B,\ B',\ B''} + 2(C,\ C',\ C'') = K \tag{2}$$

determinantis D, statuaturque

$$\begin{cases} \mathfrak{A} = B\,B - A'A'', & \mathfrak{B} = A\,B - B'B', \\ \mathfrak{A}' = B'B' - A''A, & \mathfrak{B}' = A'B' - B''B, \\ \mathfrak{A}'' = B''B'' - A\,A', & \mathfrak{B}'' = A''B'' - B\,B', \end{cases} \tag{21}$$

$$\begin{cases} \mathfrak{C} = -\mathfrak{A}\,g - \mathfrak{B}''g' - \mathfrak{B}'g'', \\ \mathfrak{C}' = -\mathfrak{B}''g - \mathfrak{A}'\,g' - \mathfrak{B}\,g'', \\ \mathfrak{C}'' = -\mathfrak{B}'\,g - \mathfrak{B}\,g' - \mathfrak{A}''g'', \\ \mathfrak{K} = K - (\mathfrak{A} + A)g\,g - (\mathfrak{A}' + A')g'g' - (\mathfrak{A}'' + A'')g''g'' \\ \quad - 2(\mathfrak{B} + B)g'g'' - 2(\mathfrak{B}' + B')g''g - 2(\mathfrak{B}'' + B'')g\,g' \\ \quad - 2Cg - 2C'g' - 2C''g'', \end{cases} \tag{22}$$

vbi g, g', g'' denotent valores (13) nunc ita exhibendos

$$(23) \qquad \begin{cases} g = -\dfrac{\mathfrak{A}\ C + \mathfrak{B}''C' + \mathfrak{B}'C''}{D}, \\ g' = -\dfrac{\mathfrak{B}''C + \mathfrak{A}'\ C' + \mathfrak{B}\ C''}{D}, \\ g'' = -\dfrac{\mathfrak{B}'\ C + \mathfrak{B}\ C' + \mathfrak{A}''C''}{D}. \end{cases}$$

Tunc noua aequatio superficiei cuiusdam secundi ordinis inter eiusdem systematis (xyz) coordinatas determinantis $\mathfrak{A}\mathfrak{B}\mathfrak{B} + \mathfrak{A}'\mathfrak{B}'\mathfrak{B}' + \mathfrak{A}''\mathfrak{B}''\mathfrak{B}'' - \mathfrak{A}\mathfrak{A}'\mathfrak{A}'' - 2\mathfrak{B}\mathfrak{B}'\mathfrak{B}'' = \mathfrak{D}$ potest componi

$$(24) \qquad \begin{pmatrix}\mathfrak{A}, & \mathfrak{A}', & \mathfrak{A}'' \\ \mathfrak{B}, & \mathfrak{B}', & \mathfrak{B}''\end{pmatrix} + 2(\mathfrak{C}, \mathfrak{C}', \mathfrak{C}'') = \mathfrak{K},$$

quam aequationi (2) *adiunctam* appellabimus.

Statim patet, aequationis ipsi (2) adiunctae coëfficientes $\mathfrak{C}$, $\mathfrak{C}'$, $\mathfrak{C}''$, $\mathfrak{K}$ indeterminatos euadere, simulatque est $D = 0$. Aequationi igitur superficiei alicuius reapse alia tamquam adiuncta respondet, si prioris determinans a cifra est diuersus, omnis autem aequatio determinantis 0 aequatione adiuncta caret.

Valor determinantis $\mathfrak{D}$ aequationis alii cuidam adiunctae inuenitur e valoribus (21), puta

$$\mathfrak{D} = (ABB + A'B'B' + A''B''B'' - AA'A'' - 2BB'B'')^2 = DD$$

siue aequalis quadrato determinantis aequationis, cui adiuncta est; ideoque determinans aequationis alii adiunctae cifrae aequare nequit ac semper esse debet quantitas positiua.

Sit inter coordinatas systematis (xyz) aequatio

$$(25) \qquad \begin{pmatrix}\mathfrak{O}, & \mathfrak{O}', & \mathfrak{O}'' \\ \mathfrak{P}, & \mathfrak{P}', & \mathfrak{P}''\end{pmatrix} + 2(\mathfrak{Q}, \mathfrak{Q}', \mathfrak{Q}'') = \mathfrak{R}$$

adiuncta ipsi (24) determinantis $\mathfrak{D}\mathfrak{D} = D^4$. Tunc manifesto protinus inuenietur

$$
(26)\qquad \begin{cases}
\mathfrak{O}\ = \mathfrak{B}\,\mathfrak{B}\ - \mathfrak{A}'\mathfrak{A}'' = A\ D, & \mathfrak{P}\ = \mathfrak{A}\ \mathfrak{B} - \mathfrak{B}'\mathfrak{B}'' = B\ D, \\
\mathfrak{O}' = \mathfrak{B}'\mathfrak{B}' - \mathfrak{A}''\mathfrak{A}\ = A'\,D, & \mathfrak{P}' = \mathfrak{A}'\,\mathfrak{B}' - \mathfrak{B}''\mathfrak{B}\ = B'\,D, \\
\mathfrak{O}'' = \mathfrak{B}''\mathfrak{B}'' - \mathfrak{A}\ \mathfrak{A}' = A''D, & \mathfrak{P}'' = \mathfrak{A}''\mathfrak{B}'' - \mathfrak{B}\ \mathfrak{B}' = B''D,
\end{cases}
$$

et, quoniam leuis calculus suppeditat

$$
(27)\qquad \begin{cases}
-\dfrac{\mathfrak{O}\ \mathfrak{C} + \mathfrak{P}''\mathfrak{C}' + \mathfrak{P}'\,\mathfrak{C}''}{\mathfrak{D}} = g\ , \\
-\dfrac{\mathfrak{P}''\mathfrak{C} + \mathfrak{O}'\mathfrak{C}' + \mathfrak{P}\ \mathfrak{C}''}{\mathfrak{D}} = g'\ , \\
-\dfrac{\mathfrak{P}'\,\mathfrak{C} + \mathfrak{P}\ \mathfrak{C}' + \mathfrak{O}''\mathfrak{C}''}{\mathfrak{D}} = g'',
\end{cases}
$$

porro esse debebit

$$
(28)\qquad \begin{cases}
\mathfrak{Q}\ = -A\ Dg - B''Dg' - B'\,Dg'', \\
\mathfrak{Q}' = -B''Dg - A'\,Dg' - B\ Dg'', \\
\mathfrak{Q}'' = -B'\,Dg - B\ Dg' - A''Dg'', \\
\mathfrak{R}\ = \mathfrak{K} - (AD + \mathfrak{A})g\,g\ - (A'D + \mathfrak{A}')g'g' - (A''D + \mathfrak{A}'')g''g'' \\
\qquad -2(BD + \mathfrak{B})g'g'' - 2(B'D + \mathfrak{B}')g''g - 2(B''D + \mathfrak{B}'')g\,g' \\
\qquad -2\mathfrak{C}g - 2\mathfrak{C}'g' - 2\mathfrak{C}''g''.
\end{cases}
$$

11.

Haud difficile est conclusu ex iis, quae artt. 9, 10 exposita sunt, aequationem (24) per transformationem primam adiumento formularum (14) earundem, quae inseruiunt ad transformationem primam aequationis (2), transire in

$$
(29)\qquad \begin{pmatrix} \mathfrak{A}, & \mathfrak{A}', & \mathfrak{A}'' \\ \mathfrak{B}, & \mathfrak{B}', & \mathfrak{B}'' \end{pmatrix} = K'
$$

variabilium x', y', z', determinantis $\mathfrak{D}$, et in qua constans K', eadem, quae in aequatione (15) obuia est, valorem induit in (16) exhibitum. Perinde aequationem (25) per transformationem primam earundem auxilio efficiendam formularum (14) manifestum est transire in

(30) $$\begin{pmatrix}\mathfrak{D}, & \mathfrak{D}', & \mathfrak{D}'' \\ \mathfrak{P}, & \mathfrak{P}', & \mathfrak{P}''\end{pmatrix} = K'$$

inter coordinatas x', y', z' determinantisque $\mathfrak{D}\mathfrak{D}$.

Iam absque vllo negotio colligi poterit, data superficiei cuiusdam secundi ordinis aequatione inter coordinatas systematis (xyz) determinantis D a cifra diuersi, infinite multas alias aequationes, inter coordinatas eiusdem illius systematis, deinceps adiunctas dari, omnesque vna cum aequatione proposita per transformationem primam ad vnum iterum commune redituras coordinatarum systema $(x'y'z')$. In schemate sequenti aequationum adiunctarum transformationem primam perpessarum vnaquaeque antecedenti est adiuncta. Indicum praefixorum 0 respondet aequationi, abs qua reliquae pendent. Adiecti sunt determinantes.

0	$\begin{pmatrix}A, & A', & A'' \\ B, & B', & B''\end{pmatrix} = K'$	D
1	$\begin{pmatrix}\mathfrak{A}, & \mathfrak{A}', & \mathfrak{A}'' \\ \mathfrak{B}, & \mathfrak{B}', & \mathfrak{B}''\end{pmatrix} = K'$	DD
2	$\begin{pmatrix}AD, & A'D, & A''D \\ BD, & B'D, & B''D\end{pmatrix} = K'$	D^4
3	$\begin{pmatrix}\mathfrak{A}DD, & \mathfrak{A}'DD, & \mathfrak{A}''DD \\ \mathfrak{B}DD, & \mathfrak{B}'DD, & \mathfrak{B}''DD\end{pmatrix} = K'$	D^8
4	$\begin{pmatrix}AD^5, & A'D^5, & A''D^5 \\ BD^5, & B'D^5, & B''D^5\end{pmatrix} = K'$	D^{16}
5	$\begin{pmatrix}\mathfrak{A}D^{10}, & \mathfrak{A}'D^{10}, & \mathfrak{A}''D^{10} \\ \mathfrak{B}D^{10}, & \mathfrak{B}'D^{10}, & \mathfrak{B}''D^{10}\end{pmatrix} = K'$	D^{32}
6	$\begin{pmatrix}AD^{21}, & A'D^{21}, & A''D^{21} \\ BD^{21}, & B'D^{21}, & B''D^{21}\end{pmatrix} = K'$	D^{64}
	etc.	
n	$\begin{pmatrix}\mathfrak{L}D^{r}, & \mathfrak{L}'D^{r}, & \mathfrak{L}''D^{r} \\ \mathfrak{M}D^{r}, & \mathfrak{M}'D^{r}, & \mathfrak{M}''D^{r}\end{pmatrix} = K'$	D^{2^n}

Pro termino generali, si n est numerus par, erit $\mathfrak{L}=A$, $\mathfrak{L}'=A'$, $\mathfrak{L}''=A''$, $\mathfrak{M}=B$, $\mathfrak{M}'=B'$, $\mathfrak{M}''=B''$, $r=1+2^2+2^4+\ldots+2^{n-2}$, si n est numerus impar, $\mathfrak{L}=\mathfrak{A}$, $\mathfrak{L}'=\mathfrak{A}'$, $\mathfrak{L}''=\mathfrak{A}''$, $\mathfrak{M}=\mathfrak{B}$, $\mathfrak{M}'=\mathfrak{B}'$, $\mathfrak{M}''=\mathfrak{B}''$, $r=2+2^3+2^5+\dots+2^{n-2}$.

12.

Quando formulae (3) eo debent inseruire, vt aequatio data superficiei certae secundi ordinis per commutanda systemata coordinatarum in alias transigatur formas, inter quantitates $\alpha, \alpha', \alpha'', \beta, \beta', \beta'', \gamma, \gamma', \gamma''$ quaspiam relationes exstare oportebit postmodum eruendas. Quoniam vero deriuatio in art. 6 facta ab huiusmodi relationibus omnino est independens, sensum formularum (3) paullulum ampliare licebit supponendo coëfficientes eorum prorsus arbitrarios. Hinc colligimus, aequationem datam $W=0$ superficiei cuiusdam F secundi ordinis per substitutionem formularum formae (3), in quibus $\alpha, \alpha', \alpha'', \beta, \beta', \beta'', \gamma, \gamma', \gamma''$, $\delta, \delta', \delta''$ sunt quantitates constantes arbitrariae, semper transire in aliam aequationem eiusdem formae $W=0$ itidem superficiem quandam G secundi ordinis repraesentantem. Denotata aequatione superficiei F per (f) superficieique G per (g), breuitatis caussa simpliciter dicemus, aequationem (f) transire in aequationem (g) vel superficiem F in superficiem G per substitutionem

$$(31)\qquad \begin{cases} x = \alpha x' + \beta y' + \gamma z' + \delta\,, \\ y = \alpha' x' + \beta' y' + \gamma' z' + \delta'\,, \\ z = \alpha'' x' + \beta'' y' + \gamma'' z' + \delta''\,, \end{cases}$$

atque (f) *implicare* ipsam (g) vel F ipsam G, siue etiam (g) sub (f), vel G sub F *contentam* esse.

Extemplo sponte patet, omnem superficiem secundi ordinis semet ipsam implicare. Data enim aequatio eam repraesentans aut reuera inuariata manebit per substitutionem (31), videlicet quum posueris $\alpha=\beta'=\gamma''=1$, $\beta=\gamma=\gamma'=\alpha'=\alpha''=\beta''=0$, $\delta=\delta'=\delta''=0$, aut in aliam repraesentantem eandem superficiem alii superstructam coordinatarum systemati axium cum eius, quod aequationi datae substernitur, parallelorum similiterque iacentium transibit per substitutionem (31), quum statueris $\alpha = \beta' = \gamma'' = 1$, $\beta = \gamma = \gamma' = \alpha' = \alpha'' = \beta'' = 0$, $\delta = d$, $\delta' = d'$ $\delta'' = d''$, vbi d, d', d'' coordinatas puncti initialis systematis posterioris relati ad primarium denotare constat.

13.

Ad instituendas quaestiones de transformandis superficiebus aequationibusque adiumento substitutionum, qualis (31), primo iam considerationes nostrae ad eas tantum superficierum secundi ordinis aequationes aliquantisper restringantur, quarum determinantes a cifra abhorreant. Quo pacto aequationes, de quibus nunc agendum erit, semper in formam

$$Axx + A'yy + A''zz + 2Byz + 2B'zx + 2B''xy = K$$

redactas supponere licet, quoniam, ni sub ista essent propositae, per transformationem primam tunc semper applicabilem ad eam reduci possent. Exinde etiam aequationes adiunctae, quibus aequationes huc pertinentes omnes debent esse praeditae, manifesto eadem vestiuntur specie.

Itaque quum in substitutione (31) feceris $\delta = \delta' = \delta'' = 0$, prodibit substitutio

$$x = \alpha x' + \beta y' + \gamma z',$$
$$y = \alpha' x' + \beta' y' + \gamma' z',$$
$$z = \alpha'' x' + \beta'' y' + \gamma'' z',$$

quam negligendo variabiles (attamen penitus intacto ordine semel inter eas stabilito) breuiter *) ita scribemus

$$(S) \qquad \begin{cases} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma''. \end{cases}$$

*) conf. Disquisitionum arithmeticarum art. 268.

Quodsi iam superficies F, quam repraesentat aequatio

$$(f) \qquad \begin{pmatrix} A, & A', & A'' \\ B, & B', & B'' \end{pmatrix} = K',$$

per substitutionem (S) transit in superficiem G, quam repraesentat aequatio

$$(g) \qquad \begin{pmatrix} L, & L', & L'' \\ M, & M', & M'' \end{pmatrix} = Q',$$

habemus ex (4)

$$(32) \quad \begin{cases} L = A\alpha\alpha + A'\alpha'\alpha' + A''\alpha''\alpha'' + 2B\alpha'\alpha'' + 2B'\alpha''\alpha + 2B''\alpha\alpha', \\ L' = A\beta\beta + A'\beta'\beta' + A''\beta''\beta'' + 2B\beta'\beta'' + 2B'\beta''\beta + 2B''\beta\beta', \\ L'' = A\gamma\gamma + A'\gamma'\gamma' + A''\gamma''\gamma'' + 2B\gamma'\gamma'' + 2B'\gamma''\gamma + 2B''\gamma\gamma', \\ M = A\beta\gamma + A'\beta'\gamma' + A''\beta''\gamma'' + B(\beta'\gamma'' + \gamma'\beta'') + B'(\beta''\gamma + \gamma''\beta) + B''(\beta\gamma' + \gamma\beta'), \\ M' = A\gamma\alpha + A'\gamma'\alpha' + A''\gamma''\alpha'' + B(\gamma'\alpha'' + \alpha'\gamma'') + B'(\gamma''\alpha + \alpha''\gamma) + B''(\gamma\alpha' + \alpha\gamma'), \\ M'' = A\alpha\beta + A'\alpha'\beta' + A''\alpha''\beta'' + B(\alpha'\beta'' + \beta'\alpha'') + B'(\alpha''\beta + \beta''\alpha) + B''(\alpha\beta' + \beta\alpha'), \\ Q' = K'. \end{cases}$$

14.

Adiumento harum aequationum deriuamus

$$LMM + L'M'M' + L''M''M'' - LL'L'' - 2MM'M'' = (ABB + A'B'B' + A''B''B'' - AA'A'' - 2BB'B'') \times$$
$$(\alpha\beta'\gamma'' + \beta\gamma'\alpha'' + \gamma\alpha'\beta'' - \gamma\beta'\alpha'' - \alpha\gamma'\beta'' - \beta\alpha'\gamma'')^2,$$

siue designato determinante aequationis (f) per D, aequationis (g) per E, quantitate reali $\alpha\beta'\gamma'' + \beta\gamma'\alpha'' + \gamma\alpha'\beta'' - \gamma\beta'\alpha'' - \alpha\gamma'\beta'' - \beta\alpha'\gamma''$ per k,

$$E = kkD.$$

Quoniam D et E a cifra discrepantes subintelligimus, erit etiam k a cifra diuersa, eruntque praediti eodem signo determinantes (a cifra diuersi) duarum aequationum, quarum vna alteram implicat.

E contemplatione aequationum (32) sponte sequitur, aequationem (f) transire in (g) etiam per hanc substitutionem

$$\begin{matrix} -\alpha, & -\beta, & -\gamma \\ -\alpha', & -\beta', & -\gamma' \\ -\alpha'', & -\beta'', & -\gamma'' \end{matrix}$$

suppeditantem loco k quantitatem $-k$, ideoque aequationem vel superficiem sub alia contentam semper ambifariam implicari *) atque superuacaneum esse, signi ipsius k peculiarem habere rationem.

Pro substitutione alia, per quam (f) transeat in (g), sit k' idem, quod k pro substitutione (S). Tunc rursus esse debebit $E = k'k'D$ adeoque $k' = k$, vnde patet, pro omnibus substitutionibus, quibus aequatio data (f) in aliam datam (g) possit transire, valorem quantitatis k eundem esse.

Quantitatem k siue $\alpha\beta'\gamma'' + \beta\gamma'\alpha'' + \gamma\alpha'\beta'' - \gamma\beta'\alpha'' - \alpha\gamma'\beta'' - \beta\alpha'\gamma''$ *normam* substitutionis vel etiam implicationis appellare licebit.

15.

Statuendo

$$(33)\quad \begin{cases} A\alpha + B''\alpha' + B'\alpha'' = [A], & B''\alpha + A'\alpha' + B\alpha'' = [A'], \quad B'\alpha + B\alpha' + A''\alpha'' = [A''], \\ A\beta + B''\beta' + B'\beta'' = [B], & B''\beta + A'\beta' + B\beta'' = [B'], \quad B'\beta + B\beta' + A''\beta'' = [B''], \\ A\gamma + B''\gamma' + B'\gamma'' = [C], & B''\gamma + A'\gamma' + B\gamma'' = [C'], \quad B'\gamma + B\gamma' + A''\gamma'' = [C''], \end{cases}$$

ex aequationibus (32) leui mutatione hae nouem eliciuntur

$$(34)\quad \begin{cases} L = \alpha[A] + \alpha'[A'] + \alpha''[A''], \\ M'' = \beta[A] + \beta'[A'] + \beta''[A''], \\ M' = \gamma[A] + \gamma'[A'] + \gamma''[A''], \\ M'' = \alpha[B] + \alpha'[B'] + \alpha''[B''], \\ L' = \beta[B] + \beta'[B'] + \beta''[B''], \\ M = \gamma[B] + \gamma'[B'] + \gamma''[B''], \\ M' = \alpha[C] + \alpha'[C'] + \alpha''[C''], \\ M = \beta[C] + \beta'[C'] + \beta'[C''], \\ L'' = \gamma[C] + \gamma'[C'] + \gamma''[C'']. \end{cases}$$

*) aliquatenus aliter res sese habet in theoria huic consimili curuarum (planarum) secundi ordinis.

Multiplicando primam, quartam, septimam per $\beta'\gamma''-\beta''\gamma'$, secundam, quintam, octauam per $\gamma'\alpha''-\gamma''\alpha'$, tertiam, sextam, nonam per $\alpha'\beta''-\alpha''\beta'$, summandoque aequationum inde profluentium, deinceps per [1], [4], [7], [2], [5], [8], [3], [6], [9] denotandarum, ternas ita, vt colligatur summa ex [1], [2], [3], summa ex [4], [5], [6] et summa ex [7], [8], [9], denique repetendo insuper bis computationem istam adhibitis loco $\beta'\gamma''-\beta''\gamma'$, $\gamma'\alpha''-\gamma''\alpha'$, $\alpha'\beta''-\alpha''\beta'$ altera vice multiplicatoribus $\beta''\gamma-\beta\gamma''$, $\gamma''\alpha-\gamma\alpha''$, $\alpha''\beta-\alpha\beta''$, ac tertia vice multiplicatoribus $\beta\gamma'-\beta'\gamma$, $\gamma\alpha'-\gamma'\alpha$, $\alpha\beta'-\alpha'\beta$, exorientur nouem aequationes, quas, designata vti supra quantitate $\alpha\beta'\gamma''+\beta\gamma'\alpha''+\gamma\alpha'\beta''-\gamma\beta'\alpha''-\alpha\gamma'\beta''-\beta\alpha'\gamma''$ per k, factisque reductionibus debitis, ita exhibeamus

$$(35)\quad \begin{cases} k[A]=L(\beta'\gamma''-\beta''\gamma')+M''(\gamma'\alpha''-\gamma''\alpha')+M'(\alpha'\beta''-\alpha''\beta'), \\ k[B]=M''(\beta'\gamma''-\beta''\gamma')+L'(\gamma'\alpha''-\gamma''\alpha')+M(\alpha'\beta''-\alpha''\beta'), \\ k[C]=M'(\beta'\gamma''-\beta''\gamma')+M(\gamma'\alpha''-\gamma''\alpha')+L''(\alpha'\beta''-\alpha''\beta'), \\ k[A']=L(\beta''\gamma-\beta\gamma'')+M''(\gamma''\alpha-\gamma\alpha'')+M'(\alpha''\beta-\alpha\beta''), \\ k[B']=M''(\beta''\gamma-\beta\gamma'')+L'(\gamma''\alpha-\gamma\alpha'')+M(\alpha''\beta-\alpha\beta''), \\ k[C']=M'(\beta''\gamma-\beta\gamma'')+M(\gamma''\alpha-\gamma\alpha'')+L''(\alpha''\beta-\alpha\beta''), \\ k[A'']=L(\beta\gamma'-\beta'\gamma)+M''(\gamma\alpha'-\gamma'\alpha)+M'(\alpha\beta'-\alpha'\beta), \\ k[B'']=M''(\beta\gamma'-\beta'\gamma)+L'(\gamma\alpha'-\gamma'\alpha)+M(\alpha\beta'-\alpha'\beta), \\ k[C'']=M'(\beta\gamma'-\beta'\gamma)+M(\gamma\alpha'-\gamma'\alpha)+L''(\alpha\beta'-\alpha'\beta). \end{cases}$$

Per algorithmum *) eundem, quo ex aequationibus (34) aequationes (35) deriuatae sunt, ex ipsis (35) nouem aliae deducuntur, quas eodem iure, quo ex (34) aequationes (32) possunt restitui, in sequentes sex contrahere licebit

$$\begin{aligned} kkA= {} & L(\beta'\gamma''-\beta''\gamma')^2+L'(\gamma'\alpha''-\gamma''\alpha')^2+L''(\alpha'\beta''-\alpha''\beta')^2 \\ & +2M(\gamma'\alpha''-\gamma''\alpha')(\alpha'\beta''-\alpha''\beta')+2M'(\alpha'\beta''-\alpha''\beta')(\beta'\gamma''-\beta''\gamma')+2M''(\beta'\gamma''-\beta''\gamma')(\gamma'\alpha''-\gamma''\alpha'), \\ kkA'= {} & L(\beta''\gamma-\beta\gamma'')^2+L'(\gamma''\alpha-\gamma\alpha'')^2+L''(\alpha''\beta-\alpha\beta'')^2 \\ & +2M(\gamma''\alpha-\gamma\alpha'')(\alpha''\beta-\alpha\beta'')+2M'(\alpha''\beta-\alpha\beta'')(\beta''\gamma-\beta\gamma'')+2M''(\beta''\gamma-\beta\gamma'')(\gamma''\alpha-\gamma\alpha''), \\ kkA''= {} & L(\beta\gamma'-\beta'\gamma)^2+L'(\gamma\alpha'-\gamma'\alpha)^2+L''(\alpha\beta'-\alpha'\beta)^2 \\ & +2M(\gamma\alpha'-\gamma'\alpha)(\alpha\beta'-\alpha'\beta)+2M'(\alpha\beta'-\alpha'\beta)(\beta\gamma'-\beta'\gamma)+2M''(\beta\gamma'-\beta'\gamma)(\gamma\alpha'-\gamma'\alpha), \end{aligned}$$

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*) Toti computo, quippe quem cl. *Seeber* fusius exposuit in opere *Untersuchungen über die Eigenschaften der positiven ternären quadratischen Formen* 1831 (pag. 37 sqq.) hic amplius immorari superfluum duximus.

$$\begin{aligned}
kkB = & \, L \, (\beta''\gamma-\beta\gamma'')(\beta\gamma'-\beta'\gamma)+L'(\gamma''\alpha-\gamma\alpha'')(\gamma\alpha'-\gamma'\alpha)+L''(\alpha''\beta-\alpha\beta'')(\alpha\beta'-\alpha'\beta) \\
& +M\,[(\gamma''\alpha-\gamma\alpha'')(\alpha\beta'-\alpha'\beta)+(\gamma\alpha'-\gamma'\alpha)(\alpha''\beta-\alpha\beta'')] \\
& +M'[(\alpha''\beta-\alpha\beta'')(\beta\gamma'-\beta'\gamma)+(\alpha\beta'-\alpha'\beta)(\beta''\gamma-\beta\gamma'')] \\
& +M''[(\beta''\gamma-\beta\gamma'')(\gamma\alpha'-\gamma'\alpha)+(\beta\gamma'-\beta'\gamma)(\gamma''\alpha-\gamma\alpha'')], \\
kkB' = & \, L \, (\beta\gamma'-\beta'\gamma)(\beta'\gamma''-\beta''\gamma')+L'(\gamma\alpha'-\gamma'\alpha)(\gamma'\alpha''-\gamma''\alpha')+L''(\alpha\beta'-\alpha'\beta)(\alpha'\beta''-\alpha''\beta') \\
& +M\,[(\gamma\alpha'-\gamma'\alpha)(\alpha'\beta''-\alpha''\beta')+(\gamma'\alpha''-\gamma''\alpha')(\alpha\beta'-\alpha'\beta)] \\
& +M'[(\alpha\beta'-\alpha'\beta)(\beta'\gamma''-\beta''\gamma')+(\alpha'\beta''-\alpha''\beta')(\beta\gamma'-\beta'\gamma)] \\
& +M''[(\beta\gamma'-\beta'\gamma)(\gamma'\alpha''-\gamma''\alpha')+(\beta'\gamma''-\beta''\gamma')(\gamma\alpha'-\gamma'\alpha)], \\
kkB'' = & \, L \, (\beta'\gamma''-\beta''\gamma')(\beta''\gamma-\beta\gamma'')+L'(\gamma'\alpha''-\gamma''\alpha)(\gamma''\alpha-\gamma\alpha'')+L''(\alpha'\beta''-\alpha''\beta')(\alpha''\beta-\alpha\beta'') \\
& +M\,[(\gamma'\alpha''-\gamma''\alpha')(\alpha''\beta-\alpha\beta'')+(\gamma''\alpha-\gamma\alpha'')(\alpha'\beta''-\alpha''\beta')] \\
& +M'[(\alpha'\beta''-\alpha''\beta')(\beta''\gamma-\beta\gamma'')+(\alpha''\beta-\alpha\beta'')(\beta'\gamma''-\beta''\gamma')] \\
& +M''[(\beta'\gamma''-\beta''\gamma')(\gamma''\alpha-\gamma\alpha'')+(\beta''\gamma-\beta\gamma'')(\gamma'\alpha''-\gamma''\alpha')].
\end{aligned}$$

E comparatione aequationum harum cum aequationibus (32) concluditur: si aequatio (f) per substitutionem (S) transit in (g), hanc ipsam (g) per substitutionem

$$(S') \qquad \begin{cases} \beta'\gamma''-\beta''\gamma', & \beta''\gamma-\beta\gamma'', & \beta\gamma'-\beta'\gamma \\ \gamma'\alpha''-\gamma''\alpha', & \gamma''\alpha-\gamma\alpha'', & \gamma\alpha'-\gamma'\alpha \\ \alpha'\beta''-\alpha''\beta', & \alpha''\beta-\alpha\beta'', & \alpha\beta'-\alpha'\beta \end{cases}$$

transire in hanc

$$(f') \qquad \begin{pmatrix} kkA, & kkA', & kkA'' \\ kkB, & kkB', & kkB'' \end{pmatrix} = K',$$

quae oritur multiplicando singulos coëfficientes partis prioris aequationis (f) per kk, siue in eandem, in quam (f) transiret per substitutionem

$$(36) \qquad \begin{cases} k, & 0, & 0 \\ 0, & k, & 0 \\ 0, & 0, & k. \end{cases}$$

Designata per F' superficie, quam repraesentat aequatio (f'), sponte patet, etiam transire superficiem G per substitutionem (S') in superficiem F'.

16.

Per calculum ei, quem art. praec. addigitauimus, haud absimilem, confirmari potest, aequationem

$$(\mathfrak{f})\qquad \begin{pmatrix}\mathfrak{A}, & \mathfrak{A}', & \mathfrak{A}''\\ \mathfrak{B}, & \mathfrak{B}', & \mathfrak{B}''\end{pmatrix} = K',$$

adiunctam ipsi (f), implicare aequationem

$$(\mathfrak{g})\qquad \begin{pmatrix}\mathfrak{L}, & \mathfrak{L}', & \mathfrak{L}''\\ \mathfrak{M}, & \mathfrak{M}', & \mathfrak{M}''\end{pmatrix} = K',$$

adiunctam ipsi (g), et in eam transire per substitutionem

$$(\mathfrak{S})\qquad \begin{cases} \beta'\gamma''-\beta''\gamma', & \gamma'\alpha''-\gamma''\alpha', & \alpha'\beta''-\alpha''\beta'\\ \beta''\gamma-\beta\gamma'', & \gamma''\alpha-\gamma\alpha'', & \alpha''\beta-\alpha\beta''\\ \beta\gamma'-\beta'\gamma, & \gamma\alpha'-\gamma'\alpha, & \alpha\beta'-\alpha'\beta. \end{cases}$$

Perinde probatu non est difficile, aequationem (g) per substitutionem

$$(\mathfrak{T})\qquad \begin{cases} \alpha, & \alpha', & \alpha''\\ \beta, & \beta', & \beta''\\ \gamma, & \gamma', & \gamma'' \end{cases}$$

transire in hanc

$$(\mathfrak{h})\qquad \begin{pmatrix}kk\mathfrak{A}, & kk\mathfrak{A}', & kk\mathfrak{A}''\\ kk\mathfrak{B}, & kk\mathfrak{B}', & kk\mathfrak{B}''\end{pmatrix} = K',$$

in eandem, in quam $(\mathfrak{f})$ transiret per substitutionem (36), siue in eam, quae oritur ex $(\mathfrak{f})$ multiplicando singulos coëfficientes partis prioris per kk. Ceterum vix opus erit monere, aequationem $(\mathfrak{h})$ ipsi (f') non esse adiunctam.

Substitutio $(\mathfrak{S})$ substitutioni (S) *adiuncta* audiat, vnde (S') adiuncta erit substitutioni $(\mathfrak{T})$. — Substitutionem $(\mathfrak{T})$ oriri dicetur per *transpositionem* substitutionis (S), tunc etiam (S) ex transpositione ipsius $(\mathfrak{T})$ atque substitutionum (S'), $(\mathfrak{S})$ vtramque ex alterius transpositione prodire patet. Protinus facile perspicietur, normas duarum substitutionum, quarum altera ex alterius transpositione oritur, aequales esse, normam autem substitutionis alii cuidam adiunctae quadratum esse normae substitutionis, cui adiuncta est. Determinantes huc requirendi ex artt. 10, 14 vltro profluunt.

Vt faciliori prouideatur conspectui, diuersae aequationum (f), (g), (f'), $(\mathfrak{f})$, $(\mathfrak{g})$, $(\mathfrak{h})$ implicationes, quas art. praes. ac praec. inuestigauimus, adiectis aequationum determinantibus substitutionumque normis tabella sequenti ante oculos ponantur.

Transeunt aequationes	Det.	in aequationes	Det.	per substitutiones	Norm.
$(f)\ldots\begin{pmatrix}A, & A', & A''\\ B, & B', & B''\end{pmatrix}=K'$	D	$(g)\ldots\begin{pmatrix}L, & L', & L''\\ M, & M', & M''\end{pmatrix}=K'$	kkD	$(S)\begin{cases}\alpha, \beta, \gamma\\ \alpha', \beta', \gamma'\\ \alpha'', \beta'', \gamma''\end{cases}$	k
$(g)\ldots\begin{pmatrix}L, & L', & L''\\ M, & M', & M''\end{pmatrix}=K'$	kkD	$(f')\ldots\begin{pmatrix}kkA, & kkA', & kkA''\\ kkB, & kkB', & kkB''\end{pmatrix}=K'$	k^6D	$(S')\begin{cases}\beta'\gamma''-\beta''\gamma', \beta''\gamma-\beta\gamma'', \beta\gamma'-\beta'\gamma\\ \gamma'\alpha''-\gamma''\alpha', \gamma''\alpha-\gamma\alpha'', \gamma\alpha'-\gamma'\alpha\\ \alpha'\beta''-\alpha''\beta', \alpha''\beta-\alpha\beta'', \alpha\beta'-\alpha'\beta\end{cases}$	kk
$(\mathfrak{f})\ldots\begin{pmatrix}\mathfrak{A}, & \mathfrak{A}', & \mathfrak{A}''\\ \mathfrak{B}, & \mathfrak{B}', & \mathfrak{B}''\end{pmatrix}=K'$	DD	$(\mathfrak{g})\ldots\begin{pmatrix}\mathfrak{L}, & \mathfrak{L}', & \mathfrak{L}''\\ \mathfrak{M}, & \mathfrak{M}', & \mathfrak{M}''\end{pmatrix}=K'$	k^4DD	$(\mathfrak{S})\begin{cases}\beta'\gamma''-\beta''\gamma', \gamma'\alpha''-\gamma''\alpha', \alpha'\beta''-\alpha''\beta'\\ \beta''\gamma-\beta\gamma'', \gamma''\alpha-\gamma\alpha'', \alpha''\beta-\alpha\beta''\\ \beta\gamma'-\beta'\gamma, \gamma\alpha'-\gamma'\alpha, \alpha\beta'-\alpha'\beta\end{cases}$	kk
$(\mathfrak{g})\ldots\begin{pmatrix}\mathfrak{L}, & \mathfrak{L}', & \mathfrak{L}''\\ \mathfrak{M}, & \mathfrak{M}', & \mathfrak{M}''\end{pmatrix}=K'$	k^4DD	$(\mathfrak{h})\ldots\begin{pmatrix}kk\mathfrak{A}, & kk\mathfrak{A}', & kk\mathfrak{A}''\\ kk\mathfrak{B}, & kk\mathfrak{B}', & kk\mathfrak{B}''\end{pmatrix}=K'$	k^6DD	$(\mathfrak{T})\begin{cases}\alpha, \alpha', \alpha''\\ \beta, \beta', \beta''\\ \gamma, \gamma', \gamma''\end{cases}$	k

17.

Substitutio (S') e substitutione (S) deriuatur per transpositionem substitutionis ipsi (S) adiunctae. Quum eadem ista ratione e substitutione $(\mathfrak{S})$ alia elicitur, prodibit

$$(\mathfrak{S}') \qquad \begin{cases}k\alpha, & k\alpha', & k\alpha''\\ k\beta, & k\beta', & k\beta''\\ k\gamma, & k\gamma', & k\gamma''\end{cases}$$

cuius norma manifesto est k^4, vti esse debet, quandoquidem ipsa substitutioni (S') adiuncta est.

Quodsi iam in tabula art. praec. loco substitutionis $(\mathfrak{T})$ hanc ipsam $(\mathfrak{S}')$ inseramus, aequatio $(\mathfrak{h})$ alii huic

$$(\mathfrak{f}') \qquad \begin{pmatrix} k^{+}\mathfrak{A}, & k^{+}\mathfrak{A}', & k^{+}\mathfrak{A}'' \\ k^{+}\mathfrak{B}, & k^{+}\mathfrak{B}', & k^{+}\mathfrak{B}'' \end{pmatrix} = K'$$

cedat oportebit determinantis $k^{12}\boldsymbol{DD}$ adeoque adiunctae aequationi (f').

Adiumento deriuationum similium duae catenae implicationum successiuarum quales tabula art. anteced. (salua tenui mutatione commodum allata) binas priores exhibet, poterunt euolui, nempe aequationum (f), (g), (f'), (g'), (f'') etc. aequationumque his deinceps adiunctarum $(\mathfrak{f})$, $(\mathfrak{g})$, $(\mathfrak{f}')$, $(\mathfrak{g}')$, $(\mathfrak{f}'')$ etc. Ac nonnullos quidem priorum artuum harum catenarum perspicuitati indulgentes hic adscribemus. Columna prima continet aequationes — primum exordientes ab ipsa (f), deinde exordientes ab $(\mathfrak{f})$ prioribus resp. adiunctas — quarum quaeque transit in proxime sequentem per substitutionem in columna tertia ambabus iunctim appositam. Singulae substitutiones seriei posterioris singulis prioris ex ordine adiunctae sunt. Columna secunda aequationum determinantes, quarta normas substitutionum exponit.

$(f)\cdots\begin{pmatrix} A, & A', & A'' \\ B, & B', & B'' \end{pmatrix}=K'$	D	$(S)\ \begin{cases} \alpha\,, & \beta\,, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{cases}$	k
$(g)\cdots\begin{pmatrix} L, & L', & L'' \\ M, & M', & M'' \end{pmatrix}=K'$	kkD	$(S')\ \begin{cases} \beta'\gamma''-\beta''\gamma', & \beta''\gamma-\beta\gamma'', & \beta\gamma'-\beta'\gamma \\ \gamma'\alpha''-\gamma''\alpha', & \gamma''\alpha-\gamma\alpha'', & \gamma\alpha'-\gamma'\alpha \\ \alpha'\beta''-\alpha''\beta', & \alpha''\beta-\alpha\beta'', & \alpha\beta'-\alpha'\beta \end{cases}$	kk
$(f')\cdots\begin{pmatrix} kkA, & kkA', & kkA'' \\ kkB, & kkB', & kkB'' \end{pmatrix}=K'$	k^6D	$(S'')\ \begin{cases} k\alpha\,, & k\beta\,, & k\gamma \\ k\alpha', & k\beta', & k\gamma' \\ k\alpha'', & k\beta'', & k\gamma'' \end{cases}$	k^4
$(g')\cdots\begin{pmatrix} k^4L, & k^4L', & k^4L'' \\ k^4M, & k^4M', & k^4M'' \end{pmatrix}=K'$	$k^{14}D$	$(S''')\ \begin{cases} kk(\beta'\gamma''-\beta''\gamma'), & kk(\beta''\gamma-\beta\gamma''), & kk(\beta\gamma'-\beta'\gamma) \\ kk(\gamma'\alpha''-\gamma''\alpha'), & kk(\gamma''\alpha-\gamma\alpha''), & kk(\gamma\alpha'-\gamma'\alpha) \\ kk(\alpha'\beta''-\alpha''\beta'), & kk(\alpha''\beta-\alpha\beta''), & kk(\alpha\beta'-\alpha'\beta) \end{cases}$	k^8
$(f'')\cdots\begin{pmatrix} k^{10}A, & k^{10}A', & k^{10}A'' \\ k^{10}B, & k^{10}B', & k^{10}B'' \end{pmatrix}=K'$	$k^{30}D$	$(S^{\text{IV}})\ \begin{cases} k^5\alpha\,, & k^5\beta\,, & k^5\gamma \\ k^5\alpha', & k^5\beta', & k^5\gamma' \\ k^5\alpha'', & k^5\beta'', & k^5\gamma'' \end{cases}$	k^{16}
$(g'')\cdots\begin{pmatrix} k^{20}L, & k^{20}L', & k^{20}L'' \\ k^{20}M, & k^{20}M', & k^{20}M'' \end{pmatrix}=K'$	$k^{62}D$	$(S^{\text{V}})\ \begin{cases} k^{10}(\beta'\gamma''-\beta''\gamma'), & k^{10}(\beta''\gamma-\beta\gamma''), & k^{10}(\beta\gamma'-\beta'\gamma) \\ k^{10}(\gamma'\alpha''-\gamma''\alpha'), & k^{10}(\gamma''\alpha-\gamma\alpha''), & k^{10}(\gamma\alpha'-\gamma'\alpha) \\ k^{10}(\alpha'\beta''-\alpha''\beta'), & k^{10}(\alpha''\beta-\alpha\beta''), & k^{10}(\alpha\beta'-\alpha'\beta) \end{cases}$	k^{32}
$(f''')\cdots\begin{pmatrix} k^{42}A, & k^{42}A', & k^{42}A'' \\ k^{42}B, & k^{42}B', & k^{42}B'' \end{pmatrix}=K'$	$k^{126}D$		
etc.		etc.	
$(\mathfrak{f})\cdots\begin{pmatrix} \mathfrak{A}, & \mathfrak{A}', & \mathfrak{A}'' \\ \mathfrak{B}, & \mathfrak{B}', & \mathfrak{B}'' \end{pmatrix}=K'$	DD	$(\mathfrak{S})\ \begin{cases} \beta'\gamma''-\beta''\gamma', & \gamma'\alpha''-\gamma''\alpha', & \alpha'\beta''-\alpha''\beta' \\ \beta''\gamma-\beta\gamma'', & \gamma''\alpha-\gamma\alpha'', & \alpha''\beta-\alpha\beta'' \\ \beta\gamma'-\beta'\gamma, & \gamma\alpha'-\gamma'\alpha, & \alpha\beta'-\alpha'\beta \end{cases}$	kk
$(\mathfrak{g})\cdots\begin{pmatrix} \mathfrak{L}, & \mathfrak{L}', & \mathfrak{L}'' \\ \mathfrak{M}, & \mathfrak{M}', & \mathfrak{M}'' \end{pmatrix}=K'$	k^4DD	$(\mathfrak{S}')\ \begin{cases} k\alpha, & k\alpha', & k\alpha'' \\ k\beta, & k\beta', & k\beta'' \\ k\gamma, & k\gamma', & k\gamma'' \end{cases}$	k^4
$(\mathfrak{f}')\cdots\begin{pmatrix} k^4\mathfrak{A}, & k^4\mathfrak{A}', & k^4\mathfrak{A}'' \\ k^4\mathfrak{B}, & k^4\mathfrak{B}', & k^4\mathfrak{B}'' \end{pmatrix}=K'$	$k^{12}DD$	$(\mathfrak{S}'')\ \begin{cases} kk(\beta'\gamma''-\beta''\gamma'), & kk(\gamma'\alpha''-\gamma''\alpha'), & kk(\alpha'\beta''-\alpha''\beta') \\ kk(\beta''\gamma-\beta\gamma''), & kk(\gamma''\alpha-\gamma\alpha''), & kk(\alpha''\beta-\alpha\beta'') \\ kk(\beta\gamma'-\beta'\gamma), & kk(\gamma\alpha'-\gamma'\alpha), & kk(\alpha\beta'-\alpha'\beta) \end{cases}$	k^8
$(\mathfrak{g}')\cdots\begin{pmatrix} k^8\mathfrak{L}, & k^8\mathfrak{L}', & k^8\mathfrak{L}'' \\ k^8\mathfrak{M}, & k^8\mathfrak{M}', & k^8\mathfrak{M}'' \end{pmatrix}=K'$	$k^{28}DD$	$(\mathfrak{S}''')\ \begin{cases} k^5\alpha, & k^5\alpha', & k^5\alpha'' \\ k^5\beta, & k^5\beta', & k^5\beta'' \\ k^5\gamma, & k^5\gamma', & k^5\gamma'' \end{cases}$	k^{16}
$(\mathfrak{f}'')\cdots\begin{pmatrix} k^{20}\mathfrak{A}, & k^{20}\mathfrak{A}', & k^{20}\mathfrak{A}'' \\ k^{20}\mathfrak{B}, & k^{20}\mathfrak{B}', & k^{20}\mathfrak{B}'' \end{pmatrix}=K'$	$k^{60}DD$	$(\mathfrak{S}^{\text{IV}})\ \begin{cases} k^{10}(\beta'\gamma''-\beta''\gamma'), & k^{10}(\gamma'\alpha''-\gamma''\alpha'), & k^{10}(\alpha'\beta''-\alpha''\beta') \\ k^{10}(\beta''\gamma-\beta\gamma''), & k^{10}(\gamma''\alpha-\gamma\alpha''), & k^{10}(\alpha''\beta-\alpha\beta'') \\ k^{10}(\beta\gamma'-\beta'\gamma), & k^{10}(\gamma\alpha'-\gamma'\alpha), & k^{10}(\alpha\beta'-\alpha'\beta) \end{cases}$	k^{32}
$(\mathfrak{g}'')\cdots\begin{pmatrix} k^{40}\mathfrak{L}, & k^{40}\mathfrak{L}', & k^{40}\mathfrak{L}'' \\ k^{40}\mathfrak{M}, & k^{40}\mathfrak{M}', & k^{40}\mathfrak{M}'' \end{pmatrix}=K'$	$k^{124}DD$	$(\mathfrak{S}^{\text{V}})\ \begin{cases} k^{21}\alpha, & k^{21}\alpha', & k^{21}\alpha'' \\ k^{21}\beta, & k^{21}\beta', & k^{21}\beta'' \\ k^{21}\gamma, & k^{21}\gamma', & k^{21}\gamma'' \end{cases}$	k^{64}
$(\mathfrak{f}''')\cdots\begin{pmatrix} k^{84}\mathfrak{A}, & k^{84}\mathfrak{A}', & k^{84}\mathfrak{A}'' \\ k^{84}\mathfrak{B}, & k^{84}\mathfrak{B}', & k^{84}\mathfrak{B}'' \end{pmatrix}=K'$	$k^{252}DD$		
etc.		etc.	

Expressiones generales, vtpote quae minus elegantes euasurae sint, quam erutu fuerint difficiliores, adiicere supersedeamus.

18.

Transeunte aequatione (f) in aequationem (g) per substitutionem

$$(S)\qquad \begin{cases} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{cases}$$

normae k, atque aequatione (g) in aliam (h) per substitutionem

$$(T)\qquad \begin{cases} \delta, & \varepsilon, & \zeta \\ \delta', & \varepsilon', & \zeta' \\ \delta'', & \varepsilon'', & \zeta'' \end{cases}$$

normae l, aequationem (f) perspicietur implicare ipsam (h) et in eam transire per substitutionem

$$(37)\qquad \begin{cases} \alpha\delta+\beta\delta'+\gamma\delta'', & \alpha\varepsilon+\beta\varepsilon'+\gamma\varepsilon'', & \alpha\zeta+\beta\zeta'+\gamma\zeta'' \\ \alpha'\delta+\beta'\delta'+\gamma'\delta'', & \alpha'\varepsilon+\beta'\varepsilon'+\gamma'\varepsilon'', & \alpha'\zeta+\beta'\zeta'+\gamma'\zeta'' \\ \alpha''\delta+\beta''\delta'+\gamma''\delta'', & \alpha''\varepsilon+\beta''\varepsilon'+\gamma''\varepsilon'', & \alpha''\zeta+\beta''\zeta'+\gamma''\zeta'' \end{cases}$$

cuius norma inuenietur

$$(\alpha\beta'\gamma''+\beta\gamma'\alpha''+\gamma\alpha'\beta''-\gamma\beta'\alpha''-\alpha\gamma'\beta''-\beta\alpha'\gamma'')(\delta\varepsilon'\zeta''+\varepsilon\zeta'\delta''+\zeta\delta'\varepsilon''-\zeta\varepsilon'\delta''-\delta\zeta'\varepsilon''-\varepsilon\delta'\zeta'')$$

i. e. $=kl$ siue aequalis producto e normis substitutionum (S) et (T). Protinus ad plures tribus aequationes perfacile ista extendetur propositio. — Ceterum hinc perleuem confirmationem expromere possemus duarum implicationum, de quibus artt. 15, 16 agebatur, scilicet secundae et quartae inter eas, quas exhibet tabella art. 16.

19.

Implicet aequatio data (determinantis a cifra diuersi) superficiem F repraesentans haec

$$(f) \qquad \begin{pmatrix} A, & A', & A'' \\ B, & B', & B'' \end{pmatrix} = K'$$

aliam (determinantis non $=0$) superficiem G repraesentantem

$$(g) \qquad \begin{pmatrix} L, & L', & L'' \\ M, & M', & M'' \end{pmatrix} = K';$$

sit (xyz) systema coordinatarum (obliquum), ad quod F refertur per (f), perinde $(x'y'z')$ systema, ad quod G refertur per (g); transeatque (f) in (g) per substitutionem

$$(S) \qquad \begin{cases} x = \alpha\, x' + \beta\, y' + \gamma\, z', \\ y = \alpha' x' + \beta' y' + \gamma' z', \\ z = \alpha'' x' + \beta'' y' + \gamma'' z' \end{cases}$$

normae k. Tunc ex substitutione (S) per eliminationem deducimus hanc nouam

$$(s) \qquad \begin{cases} x' = \dfrac{\beta'\gamma'' - \beta''\gamma'}{k} x + \dfrac{\beta''\gamma - \beta\gamma''}{k} y + \dfrac{\beta\gamma' - \beta'\gamma}{k} z, \\ y' = \dfrac{\gamma'\alpha'' - \gamma''\alpha'}{k} x + \dfrac{\alpha\gamma'' - \gamma\alpha''}{k} y + \dfrac{\gamma\alpha' - \gamma'\alpha}{k} z, \\ z' = \dfrac{\alpha'\beta'' - \alpha''\beta'}{k} x + \dfrac{\alpha''\beta - \alpha\beta''}{k} y + \dfrac{\alpha\beta' - \alpha'\beta}{k} z \end{cases}$$

normae $\frac{1}{k}$, per quam aequatio (g) redire debet in aequationem (f).

Omnis igitur aequatio ab alia implicata ipsa eam implicat, sub qua contenta est, idemque valet de superficiebus.

Substitutiones (S) et (s), quarum altera ab (f) ad (g), altera a (g) ad (f) instituit transgressum, atque implicationes inde demanantes *reciprocas* nominare conueniet. Manifestum est, productum e normis substitutionum reciprocarum semper vnitati positiuae reali aequare.

Aequationem $(\mathfrak{f})$ ipsi (f) adiunctam in aequationem $(\mathfrak{g})$ ipsi (g) adiunctam notum est transire per substitutionem

$$(\mathfrak{S})\quad \begin{cases} \beta'\gamma''-\beta''\gamma', & \gamma'\alpha''-\gamma''\alpha', & \alpha'\beta''-\alpha''\beta' \\ \beta''\gamma-\beta\gamma'', & \gamma''\alpha-\gamma\alpha'', & \alpha''\beta-\alpha\beta'' \\ \beta\gamma'-\beta'\gamma, & \gamma\alpha'-\gamma'\alpha, & \alpha\beta'-\alpha'\beta \end{cases}$$

normae kk. Hinc prodit substitutio reciproca ipsius $(\mathfrak{S})$ puta

$$(\mathfrak{s})\quad \begin{cases} \frac{\alpha}{k}, & \frac{\alpha'}{k}, & \frac{\alpha''}{k} \\ \frac{\beta}{k}, & \frac{\beta'}{k}, & \frac{\beta''}{k} \\ \frac{\gamma}{k}, & \frac{\gamma'}{k}, & \frac{\gamma''}{k} \end{cases}$$

normae $\frac{1}{kk}$, per quam $(\mathfrak{g})$ regredietur ad aequationem $(\mathfrak{f})$.

20.

Duae aequationes sese inuicem implicantes, inter quas transitus efficiuntur per substitutiones reciprocas normarum aequalium, *aequiualentes* dicentur.

Statim patet, eiusmodi substitutionum normam semper esse ± 1, ideoque aequationes aequiualentes determinantibus aequalibus gaudere (vid. art. 14), necnon aequationes aequiualentibus adiunctas ipsas esse aequiualentes.

E duabus implicationum successiuarum seriebus art. 17 expositis elucebit, hoc nostro casu, quo fit $k = \pm 1$, aequationes (f'), (f''), (f''') etc. identicas euadere cum ipsa (f), et (g'), (g''), (g''') etc. cum (g), ac proinde $(\mathfrak{f}')$, $(\mathfrak{f}'')$, $(\mathfrak{f}''')$ etc. cum $(\mathfrak{f})$, et $(\mathfrak{g}')$, $(\mathfrak{g}'')$, $(\mathfrak{g}''')$ etc. cum $(\mathfrak{g})$, denique vero etiam substitutiones (S''), (S^{IV}), (S^{VI}) etc. cum ipsa (S); (S'''), (S^{V}) etc. cum (S'), et $(\mathfrak{S}'')$, $(\mathfrak{S}^{IV})$, $(\mathfrak{S}^{VI})$ etc. cum $(\mathfrak{S})$; $(\mathfrak{S}''')$, $(\mathfrak{S}^{V})$ etc. cum $(\mathfrak{S}')$. Vnde porro patebit, substitutionum ambarum

$$(S)\quad \begin{cases} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{cases}$$

normae ± 1, per quam transit $(f) \cdots \begin{pmatrix} A, A', A'' \\ B, B', B'' \end{pmatrix} = K'$ in aequiualentem $(g) \cdots \begin{pmatrix} L, L', L'' \\ M, M', M'' \end{pmatrix} = K'$ et

$$(S') \qquad \begin{cases} \beta'\gamma'' - \beta''\gamma', & \beta''\gamma - \beta\gamma'', & \beta\gamma' - \beta'\gamma \\ \gamma'\alpha'' - \gamma''\alpha', & \gamma''\alpha - \gamma\alpha'', & \gamma\alpha' - \gamma'\alpha \\ \alpha'\beta'' - \alpha''\beta', & \alpha''\beta - \alpha\beta'', & \alpha\beta' - \alpha'\beta \end{cases}$$

normae ± 1 *), per quam transit (g) in aequiualentem (f), vtramque exoriri per transpositionem substitutionis alteri adiunctae. Idem valet de substitutionibus $(\mathfrak{S})$, $(\mathfrak{S}')$.

21.

Ex data qualibet substitutione

$$(S) \qquad \begin{cases} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{cases}$$

normae k non $= \pm 1$ alia perfacile exstruitur normae ± 1 et coëfficientium coëfficientibus ipsius (S) resp. proportionalium. Talis enim erit

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*) Proprie quidem implicationis tantum norma reuera est ± 1, i. e. tum $+1$ quum -1. Vbi de substitutionibus sermo est, aliter quodammodo res se habet. Etenim substitutionis (S) norma aut est $+1$ aut -1; quodsi est $+1$, alia datur (conf. art. 14) eodem vice fungens, cui est -1; sin -1, alia, cui $+1$. Atqui quatenus forma generalis inuoluat casus ambos, impune dicebis, substitutionis (S) normam esse ± 1. Sed quoad substitutionem (S'), ni placuerit caute scribere

$$\begin{matrix} \pm \beta'\gamma'' \mp \beta''\gamma', & \pm \beta''\gamma \mp \beta\gamma'', & \pm \beta\gamma' \mp \beta'\gamma \\ \pm \gamma'\alpha'' \mp \gamma''\alpha', & \pm \gamma''\alpha \mp \gamma\alpha'', & \pm \gamma\alpha' \mp \gamma'\alpha \\ \pm \alpha'\beta'' \mp \alpha''\beta', & \pm \alpha''\beta \mp \alpha\beta'', & \pm \alpha\beta' \mp \alpha'\beta, \end{matrix}$$

locutio nostra et hocce loco et in posterum per synesin intelligenda est, quoniam substitutioni isti in forma (S'), vti leuis docet attentio, semper est norma positiua, sin mutaueris signa, semper negatiua.

$$(S^0)\qquad \begin{cases} \dfrac{\alpha}{k^{\frac{1}{3}}}, & \dfrac{\beta}{k^{\frac{1}{3}}}, & \dfrac{\gamma}{k^{\frac{1}{3}}} \\ \dfrac{\alpha'}{k^{\frac{1}{3}}}, & \dfrac{\beta'}{k^{\frac{1}{3}}}, & \dfrac{\gamma'}{k^{\frac{1}{3}}} \\ \dfrac{\alpha''}{k^{\frac{1}{3}}}, & \dfrac{\beta''}{k^{\frac{1}{3}}}, & \dfrac{\gamma''}{k^{\frac{1}{3}}}, \end{cases}$$

vbi, siquidem e contemplationibus nostris coëfficientes imaginarios excludere nobis sit propositum, per expressiones radicales solos valores reales intelligi oportebit.

Quo pacto, si per substitutionem (S) transit aequatio (f) in (g), transire debebit per (S^0) aequatio (f) in aliam ipsi aequiualentem, puta in eam, quae prodit ex (g) diuidendo singulos partis prioris coëfficientes per $k^{\frac{2}{3}}$. Substitutio reciproca ipsius (S^0) deducitur e substitutione (S') diuidendo singulos coefficientes per $k^{\frac{2}{3}}$, vel ex ipsa (s) multiplicando singulos coëfficientes per $k^{\frac{1}{3}}$. Tali modo reductio casus praesentis, vt substitutionis propositae norma ab vnitate reali (seu positiua seu negatiua) discrepet, ad casum praecedentem, quo norma est $= \pm 1$, absque vllo negotio in totas implicationum schemate art. 17 expositarum series expandi poterit.

22.

Implicationum hucusque analytice disquisitarum significationes geometricae in commentatione hancce insecutura explorandae proxime sese offerent, protinusque elucebit, quoad determinantes a cifra diuersos, superficies, quarum aequationes implicationibus inter se cohaerent, similes, quarum aequationes sunt aequiualentes, aequales esse. Instituta exinde transformatione secunda, qua $\begin{pmatrix} A, & A', & A'' \\ B, & B', & B'' \end{pmatrix} = K'$ transgeretur ad aequiualentem $\begin{pmatrix} L, & L', & L'' \\ 0, & 0, & 0 \end{pmatrix} = K'$, propulsaque perquisitione ad

(S^0)

vbi, siquidem e contempl [illegible] inarios excludere nobis sit propositum, per expre [illegible] s intelligi oportebit.

Quo pacto, si pe [illegible] o (f) in (g), transire debebit per (S^0) aequatio [illegible], puta in eam, quae prodit ex (g) diuidendo [illegible] per $k^{\frac{2}{3}}$. Substitutio reciproca ipsius (S^0) dedu [illegible] o singulos coefficientes per $k^{\frac{2}{3}}$, vel ex ipsa (s) [illegible] per $k^{\frac{1}{3}}$. Tali modo reductio casus praesentis, [illegible] ab vnitate reali (seu positiua seu negatiua) disc [illegible] o norma est $= \pm 1$, absque vllo negotio in t [illegible] 17 expositarum series expandi poterit.

Implicationum hucu [illegible] ficationes geometricae in commentatione hancce in [illegible] offerent, protinusque elucebit, quoad determinan [illegible] quarum aequationes implicationibus inter se coh [illegible] es sunt aequiualentes, aequales esse. Instituta exi [illegible] $\begin{pmatrix}A, & A', & A''\\ B, & B', & B''\end{pmatrix} = K'$ transgeretur ad aequiualente [illegible] que perquisitione ad tales quoque aequationes vel (vti tunc loquemur) superficies, quarum determinantes cifrae sint aequales, ad determinationem axium principalium via erit aperta. **Particula** tertia sequentesque singulis superficierum secundi ordinis generibus et speciebus inuestigandis necnon quaestionibus quibusdam tummaxime emersuris dicatae erunt.