Quaedam ad integrationem functionis differentialis

$\frac{\varphi(x) \, . \, dx}{\sqrt{(a+bx+cx^2)}}$ *pertinentia*

scripsit

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I.

In melioris notae omnibus libris, quibus integrandi praecepta conscripta sunt, ratio traditur, quae ingredienda sit ad tractandas functiones

$$\int \varphi x \sqrt{(a+bx+cx^2)}dx; \int \frac{\varphi x.\, dx}{\sqrt{(a+bx+cx^2)}}$$

quarum altera facile multiplicatione et divisione quantitatis $\sqrt{(a+bx+cx^2)}$ in alteram transformari potest. Ut exemplis utar, invenies artificia integrationis necessaria in enchiridio a Lacroix conscripto T. II, §. 183, sqq. ed. Baumann; a Navier conscripto T. I. §. 274, sqq. ed. H. Wittstein, 1854; in lexico d. Kluegelii s. v. Jntegratio; denique in d. Crellii ephemeridibus T. V. Quibus in nostrum usum conversis rebus quasdam argumentationes et consectaria addere mihi in animo est. Rationes integrationis hoc modo proponimus: Sit primo quantitas c positiva, tum poterimus has duas transformationes adhibere, quibus functio rationalis et ad integrandum apta evadat:

1. Faciamus $\sqrt{(a+bx+cx^2)} = z-x\sqrt{c}$ ubi z est nova quantitas variabilis, quae ex functione quantitatis x pendeat. Jnde eruendo valores quantitatum x et dx, invenimus:

$$x=\frac{z^2-a}{b+2z\sqrt{c}};\ dx=\frac{2(bz+z^2\sqrt{c}+a\sqrt{c})}{(b+2z\sqrt{c})^2}dz;\ \sqrt{(a+bx+cx^2)}=\frac{bz+z^2\sqrt{c}+a\sqrt{c}}{b+2z\sqrt{c}}$$

quibus substitutis reperimus

$$\int \frac{\varphi x.\, dx}{\sqrt{(a+bx+cx^2)}} = \int \varphi \left\{\frac{z^2-a}{b+2\sqrt{c}.z}\right\} \frac{2dz}{b+2\sqrt{c}.z}$$

Tali modo rationalis facta integrari poterit, qua integratione absoluta, substitues

$$z=\sqrt{(a+bx+cx^2)}+x\sqrt{c}.$$

Annotandum est, eodem modo ex functione $\frac{z^2-a}{b+2\sqrt{c}.z}$ formandam esse functionem $\varphi\left\{\frac{z^2-a}{b+2\sqrt{c}.z}\right\}$, quo φx ex variabili x efficta sit, neque signum φ pertinere nisi ad uncis inclusam quantitem proxime sequentem.

2. Quod si pones $\sqrt{(a+bx+cx^2)} = z+x\sqrt{c}$, eadem via invenies

$$\int \frac{\varphi x.\, dx}{\sqrt{(a+bx+cx^2)}} = \int \varphi \left\{\frac{a-z^2}{2z\sqrt{c}-b}\right\} . \frac{-2dz}{2z\sqrt{c}-b}$$ ubi post integrationem facies $z=\sqrt{(a+bx+cx^2)}-x\sqrt{c}$.

Transeamus jam ad casum, ubi c est quantitas negativa; ad integrandam functionem $\frac{\varphi x.\ dx}{\sqrt{(a+bx-cx^2)}}$ fac: $\sqrt{(a+bx-cx^2)} = xz-\sqrt{a}$, unde differentiando et substituendo habebis

$$3.\quad \int \frac{\varphi x.\ dx}{\sqrt{(a+bx-cx^2)}} = \int \varphi \left\{ \frac{b+2z\sqrt{a}}{c+z^2} \right\} . \frac{-2dz}{c+z^2}$$

qua in formula integratione absoluta substituito $z = \frac{\sqrt{(a+bx-cx^2)}+\sqrt{a}}{x}$. Aeque recte res sese habebit, si facies $\sqrt{(a+bx-cx^2)} = xz+\sqrt{a}$, unde obtinebis

$$4.\quad \int \frac{\varphi x.\ dx}{\sqrt{(a+bx-cx^2)}} = \int \varphi \left\{ \frac{b-2z\sqrt{a}.}{c+z^2} \right\} . \frac{-2dz}{c+z^2}$$ et invento integrali substitues

$$z = \frac{\sqrt{(a+bx-cx^2)}-\sqrt{a}.}{x}.$$

Si tibi displicebit, transformatione modo exposita introduci quantitatem imaginariam tum, quum a est negativa, evitabis id incommodum dispescendo $a+bx-cx^2$ in factores reales $c\,(x-R)\,(r-x)$ quae res, qui fiat, facillime invenitur. Quod cum obtinueris $\sqrt{(a+bx-cx^2)} = \sqrt{[c(x-R)\,(r-x)]}$, fac $\sqrt{[c(x-R)\,(r-x]} = (x-R)z\sqrt{c}$ et cum expresseris x, dx, $\sqrt{(a+bx-cx^2)}$ ope functionum variabilis quantitatis z, obtinebis

$$5.\quad \int \frac{\varphi x.\ dx}{\sqrt{(a+bx-cx^2)}} = \int \varphi \left\{ \frac{r+Rz^2}{1+z^2} \right\} . \frac{-2dz}{(1+z^2)\sqrt{c}}$$ qua integratione absoluta substitues $z = \sqrt{\frac{r-x}{x-R}}$.

Jam ad applicationes legum explicatarum progrediamur, quae sunt uberrimi argumenti.

II.

Primum accommodemus inventa ad functionem $\int \frac{d_x}{\sqrt{(1-x^2)}}$ ponamus igitur, comparatione cum generali functione facta, $\varphi x = 1$; $a = +1$; $b = o$; $-c = -1$, inveniemus has quinque formulas:

$$1.\quad \int \frac{dx}{\sqrt{(1-x^2)}} = \int \frac{dz}{z\sqrt{-1}} \quad (\text{cond: } z = \sqrt{(1-x^2)}+x\sqrt{-1})$$

$$= C + \frac{1}{\sqrt{-1}} \log \left\{ x\sqrt{-1} + \sqrt{(1-x^2)} \right\}$$

$$2.\quad \int \frac{dx}{\sqrt{(1-x^2)}} = \int \frac{-dz}{z\sqrt{-1}} \quad (\text{cond: } z = \sqrt{(1-x^2)}-x\sqrt{-1})$$

$$= C - \frac{1}{\sqrt{-1}} \log \left\{ -x\sqrt{-1} + \sqrt{(1-x^2)} \right\}$$

3. $\int\frac{dx}{\sqrt{(1-x^2)}}=\int\frac{-2dz}{1+z^2}$ (cond: $z=\frac{\sqrt{(1-x^2)}+1}{x}$)

$=C-2\ \text{arc}\ (\text{tng}=\frac{1+\sqrt{(1-x^2)}}{x})$

4. $\int\frac{dx}{\sqrt{(1-x^2)}}-\int\frac{-2dz}{1+z^2}$ (cond: $z=\frac{\sqrt{(1-x^2)}-1}{x}$

$=C-2\ \text{arc}\ (\text{tng}=\frac{\sqrt{(1-x^2)}-1}{x})$

5. $\int\frac{dx}{\sqrt{(1-x^2)}}=\int\frac{-2dz}{1+z^2}$ (condd: $r=1$; $R=-1$; $z=\sqrt{\frac{1-x}{1+x}}$)

$=C-2\ \text{arc}\ (\text{tng}=\sqrt{\frac{1-x}{1+x}})$

Jam vides quinque integrationes, praeter quas duae aliae notissimae sunt. Etenim si valorem imaginarium denominatoris $\sqrt{(1-x^2)}$ evitare vis, supponere debes, esse $x<1$; si $x>1$ est, mutabis $\sqrt{(1-x^2)}$ in $\sqrt{-1}\ .\ \sqrt{x^2-1}$, qui casus paulo post tractabitur. — Quod si igitur $x<1$ est, theorema binomiale seriem convergentem dabit, si adhibueris id ad evolvendam functionem $(1-x^2)^{-\frac{1}{2}}$, qua multiplicata cum dx et membratim integrata obtinebis $\int\frac{dx}{\sqrt{(1-x^2)}}=C+x+\frac{1}{2}.\frac{x^3}{3}+\frac{1.\ 3}{2.\ 4}.\frac{x^5}{5}+\frac{1.\ 3.\ 5}{2.\ 4.\ 6}.\frac{x^7}{7}+\ldots\ldots$

Praeterea constat, esse $\int\frac{dx}{\sqrt{(1-x^2)}}=C+\text{arc}\ (\sin=x)$ Si his duobus in valoribus posueris $x=0$, videbis utrimque constantem numerum eundem esse, unde sub conditione, ut sit $x\leqq 1$ et omnium arcuum minimum sumas, concludes:

$$\text{arc}\ (\sin=x)=x+\frac{1}{2}.\frac{x^3}{3}+\frac{1.\ 3}{2.\ 4}.\frac{x^5}{5}+\frac{1.\ 3.\ 5}{2.\ 4.\ 6}.\frac{x^7}{7}+\ldots\ldots$$

$$\text{arc}\ (\sin=1)=\frac{\pi}{2}=1+\frac{1}{2}.\frac{1}{3}+\frac{1.\ 3}{2.\ 4}.\frac{1}{5}+\frac{1.\ 3.\ 5}{2.\ 4.\ 6}.\frac{1}{7}+\ldots\ldots$$

$$\text{arc}\ (\sin=\tfrac{1}{2})=\frac{\pi}{6}=\frac{1}{2}+\frac{1}{2}.\frac{1}{3}.\frac{1}{2^3}+\frac{1.\ 3}{2.\ 4}.\frac{1}{5.2^5}+\frac{1.\ 3.\ 5}{2.\ 4.\ 6}.\frac{1}{7.2^7}+\ldots\ldots$$

Quod si haec integralia cum quinque sub numeris 1), 2), 3), 4), 5) propositis conferes, videbis, in formulis 1) et 2) numerum constantem eundem esse, cum identidem pro valore $x=0$ sit $\int\frac{dx}{\sqrt{(1-x^2)}}=C$, unde invenies, si arc $(\sin=x)=\varphi$ ponatur, formulas, quas tanquam fundamentum totius doctrinae functionum cyclicarum esse constat, nempe:

$\varphi\sqrt{-1}=\log\ (x\sqrt{-1}+\sqrt{[1-x^2]})$

$-\varphi\sqrt{-1}=\log\ (-x\sqrt{-1}+\sqrt{[1-x^2]})$

Transeundo a logarithmis naturalibus ad numeros congruentes et ab aequatione arc $(\sin=x)=\varphi$

ad conversam sin $\varphi = x$; cos $\varphi = \sqrt{[1-x^2]}$ habebimus $e^{\varphi\sqrt{-1}} = \cos\varphi + \sin\varphi\sqrt{-1}$;
$e^{-\varphi\sqrt{-1}} = \cos\varphi - \sin\varphi\sqrt{-1}$ unde deducimus $\cos\varphi = \frac{1}{2}\left\{e^{\varphi\sqrt{-1}} + e^{-\varphi\sqrt{-1}}\right\}$
$\sin\varphi = \frac{1}{2\sqrt{-1}}\left\{e^{\varphi\sqrt{-1}} - e^{-\varphi\sqrt{-1}}\right\}$; et theorema Moivricum omnium memoratu dignissimum et universale $(\cos\varphi \pm \sin\varphi\sqrt{-1})^n = \cos n\varphi \pm \sin n\varphi.\sqrt{-1}$

Ubi hactenus pervenisti, tota doctrina de functionibus cyclicis secundo vento procedit, quare nihil addendum puto, cum cuivis Mathematicorum aliquatenus perito deductio ceterorum theorematum res trita sit. Notatu dignum igitur tantum videtur, pro valore $\varphi = \pi$ esse $\cos\frac{\pi}{2} = 0$, $\sin\frac{\pi}{2} = 1$, unde eo in casu deduci ex formula $\varphi\sqrt{-1} =$

$\log(\cos\varphi + \sin\varphi\sqrt{-1})$ specialem $\frac{\pi}{2}\sqrt{-1} = \log\sqrt{-1}$, eoque effici

$\pi = \frac{2\log\sqrt{-1}}{\sqrt{-1}}$. Jam memento, esse $\sqrt{-1}.[1-\sqrt{-1}] = \sqrt{-1}+1 = 1+\sqrt{-1}$, ergo $\log\sqrt{-1} = \log\frac{1+\sqrt{-1}}{1+\sqrt{-1}} = \log(1+\sqrt{-1}) - \log(1-\sqrt{-1})$. Evolvendo utrumque logarithmum ope formulae notissimae $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$ et subtrahendo habebis $2(\sqrt{-1} - \frac{1}{3}\sqrt{-1} + \frac{1}{5}\sqrt{-1} - \frac{1}{7}\sqrt{-1} \ldots\ldots\ldots)$ unde $\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \ldots\ldots)$. Demonstrari potest, hanc seriem, etsi lentissime, attamen convergere, ejusque summa methodi Huttonianae ope facillime inveniri potest. — Transeamus ad tertium integrale

$$\int\frac{dx}{\sqrt{1-x^2}} = C - 2\,\mathrm{arc}\left(\mathrm{tng} = \frac{1+\sqrt{[1-x^2]}}{x}\right)$$ et faciamus $2\,\mathrm{arc}\left(\mathrm{tng} = \frac{1+\sqrt{1-x^2}}{x}\right) = \varphi$, ergo $\mathrm{arc}\left(\mathrm{tng} = \frac{1+\sqrt{[1-x^2]}}{x}\right) = \frac{\varphi}{2}$; unde $\mathrm{tng}\frac{\varphi}{2} = \frac{1+\sqrt{[1-x^2]}}{x}$

$x\,\mathrm{tng}\frac{\varphi}{2} - 1 = \sqrt{[1-x^2]}$; $x^2\,\mathrm{tng}^2\frac{\varphi}{2} - 2x\,\mathrm{tng}\frac{\varphi}{2} + 1 = 1 - x^2$; $x^2(1 + \mathrm{tng}^2\frac{\varphi}{2}) = 2\,\mathrm{tng}\frac{\varphi}{2}$;

$$\frac{x}{\cos^2\frac{\varphi}{2}} = \frac{2\sin\frac{\varphi}{2}}{\cos\frac{\varphi}{2}}$$

$x = 2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} = \sin\varphi = \sin(180^0 - \varphi)$; $-\varphi = -180^0 + \mathrm{arc}(\sin = x)$;

$$\int\frac{dx}{(\sqrt{1-x^2})} = C - 180^0 + \mathrm{arc}(\sin = x)$$
$$= C^1 + \mathrm{arc}(\sin = x)$$

Ponamus eodem modo in quarto integrali

$$\int \frac{dx}{\sqrt{[1-x^2]}} = C - 2 \text{ arc } \left(\text{tng} = \frac{\sqrt{[1-x^2]}-1}{x}\right)$$

$$2 \text{ arc } \left(\text{tng} = \frac{\sqrt{[1-x^2]}-1}{x}\right); \text{ et ex eo deducimus tng } \frac{\varphi}{2} = \frac{\sqrt{[1-x^2]}-1}{x};$$

$$1 + x\text{tng}\frac{\varphi}{2} = \sqrt{[1-x^2]};\ 1 + 2x\,\text{tng}\,\frac{\varphi}{2} + x^2\,\text{tng}^2\frac{\varphi}{2} = 1 - x^2;$$

$$2x\,\frac{\sin\frac{\varphi}{2}}{\cos\frac{\varphi}{2}} + x^2\left\{\frac{1}{\cos^2\frac{\varphi}{2}}\right\} = 0;\ 2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} + x = 0;\ x = -\sin\varphi = \sin(-\varphi);$$

$-\varphi = \text{arc } (\sin = x)$; unde concludimus

$$\int \frac{dx}{\sqrt{[1-x^2]}} = C - 2 \text{ arc } \left(\text{tng} = \frac{\sqrt{[1-x^2-1]}}{x}\right)$$
$$= C + \text{arc } (\sin = x).$$

Si transitur ad quintum integrale

$$\int \frac{dx}{\sqrt{[1-x^2]}} = C - 2 \text{ arc } \left(\text{tng} = \sqrt{\frac{1-x}{1+x}}\right)$$

facimus $\text{arc } \left(\text{tng} = \sqrt{\frac{1-x}{1+x}}\right) = \frac{\varphi}{2}$; unde concludimus $\text{tng}^2\,\frac{\varphi}{2} = \frac{1-x}{1+x}$

$$\text{tng}^2\frac{\varphi}{2} + x\,\text{tng}^2\,\frac{\varphi}{2} = 1 - x;\ x + x\,\text{tng}^2\frac{\varphi}{2} + \text{tng}^2\,\frac{\varphi}{2} = 1;\ \frac{x}{\cos^2\frac{\varphi}{2}} + \text{tng}^2\frac{\varphi}{2} = 1;$$

$$x + \sin^2\frac{\varphi}{2} = \cos^2\frac{\varphi}{2};\ x = \cos^2\frac{\varphi}{2} - \sin^2\frac{\varphi}{2} = \cos\varphi = \sin\left(\frac{\pi}{2} - \varphi\right);$$

$-\varphi = -\frac{\pi}{2} + \text{arc } (\sin = x)$; et denique

$$\int \frac{dx}{\sqrt{[1-x^2]}} = C - \frac{\pi}{2} + \text{arc } (\sin = x)$$
$$= C^1 + \text{arc } (\sin = x).$$

III.

Pergamus ad integrandam functionem

$$\int \frac{dx}{\sqrt{(1+x^2)}}$$

ad quam ut applicemus formulas generales, faciendum in his erit $\varphi(x) = 1$, $a = 1$, $b = 0$, $c = +1$, quibus valoribus introductis prodibunt hae functiones non minus tractatis fertiles: Secundum normam primam et secundam capitis I. invenimus:

1. $$\int \frac{dx}{\sqrt{[1+x^2]}} = \int \frac{dz}{z} = C^1 + \log\left\{\sqrt{[1+x^2]} + x\right\}$$

2. $\int \frac{dx}{\sqrt{[1+x^2]}} = \int -\frac{dz}{z} = C^1 - \log \left\{ \sqrt{[1+x^2]} - z \right\}$

Jn utroque integrali numerus constans ejusdem est valoris, cum pro $x=0$ functio logarithmica sit $= 0$, quare, si ponimus $\log (\sqrt{[1+x^2]}+x) = \psi$, etiam erit $-\log (\sqrt{[1+x^2]}-x) = +\psi$, unde transeundo ex logarithmis ad numeros congruentes efficietur, ut sit

$e^{\psi} = \sqrt{[1+x^2]} + x$; $e^{-\psi} = \sqrt{[1+x^2]} - x$; ergo addendo et subtrahendo

$$\sqrt{[1+x^2]} = \frac{e^{\psi} + e^{-\psi}}{2}$$

$$x = \frac{e^{\psi} - e^{-\psi}}{2}$$

Jam videmus, similitudinem maximam cum functionibus cyclicis existere, unde eligendam esse formulam et signa congruentia. Faciamus igitur

$\mathfrak{Sin}\ \psi = x$; et vicissim ψ — $\mathfrak{Arc}\ (\mathfrak{Sin}=x)$; inde deducemus

$\mathfrak{Sin}\ \psi = \frac{e^{\psi} - e^{-\psi}}{2}$; $\mathfrak{Sin}\ \psi = \frac{\psi}{1} + \frac{\psi^3}{1.2.3.} + \frac{\psi^5}{1.2.3.4.5.} + \ldots\ldots$

$\sqrt{(1 + \mathfrak{Sin}^2 \psi)} = \sqrt{[1+x^2]} = \mathfrak{Cos}\ \psi$; $\mathfrak{Cos}\ \psi = \frac{e^{\psi} + e^{-\psi}}{2}$;

$\mathfrak{Cos}\ \psi = 1 + \frac{\psi^2}{1.2} + \frac{\psi^4}{1.2.3.4} + \ldots\ldots$; $\psi = \mathfrak{Arc}\ (\mathfrak{Cos} = \sqrt{[1+x^2]})$

$\mathfrak{Tng}\ \psi = \frac{\mathfrak{Sin}\ \psi}{\mathfrak{Cos}\ \psi}$; $\mathfrak{Cot}\ \psi = \frac{\mathfrak{Cos}\ \psi}{\mathfrak{Sin}\ \psi} = \frac{1}{\mathfrak{Tng}\ \psi}$

Evolutae formulae jam sufficiunt ad formandam doctrinam totam horum sinuum, qui hyperbolici vocantur, quoniam eodem fere modo cum hyperbola cohaerent, quo cyclici cum communi cyclo. Jnitium doctrinae ejus generis efficit, quod jam invenimus

$$\int \frac{dx}{\sqrt{[1+x^2]}} = C + \mathfrak{Arc}\ (\mathfrak{Sin} = x)$$

sive in seriem evolvendo et membratim integrando

$$\int \frac{dx}{\sqrt{[1+x^2]}} = C + \frac{x^3}{2.3} + \frac{1.3}{2.4}\frac{x^5}{5} - \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \ldots\ldots$$

$$= C \pm \log \left\{ \sqrt{[1+x^2]} \pm x \right\}$$

Adhibeamus tertiam formulam generalem, ex qua sequitur, ut sit

3. $\int \frac{dx}{\sqrt{[1+x^2]}} = \int \frac{-2dz}{z^2-1}$ (cond: $\frac{\sqrt{[1+x^2]} + 1}{x} = z$)

Ad inveniendum hoc integrale ponimus primo

$$\mathfrak{Cos}^2 u - \mathfrak{Sin}^2 u = \frac{\{e^{u} + e^{-u}\}^2}{4} - \frac{\{e^{u} - e^{-u}\}^2}{4} = 1;$$

quare dividendo habebimus

$\frac{\mathfrak{Cos}^2 u}{\mathfrak{Sin}^2 u} - 1 = \frac{1}{\mathfrak{Sin}^2 u}$; $\mathfrak{Cot}^2 u - 1 = \frac{1}{\mathfrak{Sin}^2 u}$; Si autem est $\mathfrak{Cot}\, u = z$; ergo $u = \mathfrak{Arc}\,(\mathfrak{Cot} = z)$; inde concludimus $dz = d\,\mathfrak{Cot}\,u = d\frac{\mathfrak{Cos}\,u}{\mathfrak{Sin}\,u} = \frac{-du}{\mathfrak{Sin}^2 u}$;

$= -du\,(z^2-1)$; $du = \frac{-dz}{z^2-1}$ ergo

$$\int \frac{-dz}{z^2-1} = \int du = C + \mathfrak{Arc}\,(\mathfrak{Cot} = z); \text{ unde}$$

$$\int \frac{2dz}{z^2-1} = C + 2\,\mathfrak{Arc}\,(\mathfrak{Cot} = z); \; z = \frac{\sqrt{[1+x^2]}+1}{x}$$

Ut autem hoc integrale ad inventam sinus hyperbolici formulam reducatur, faciamus

$2\,\mathfrak{Arc}\,(\mathfrak{Cot} = z)\,\varphi$; $\mathfrak{Arc}\,(\mathfrak{Cot} = \frac{\sqrt{[1+x^2]}+1}{x} = \frac{\varphi}{2}$;) ergo est $\mathfrak{Cot}\frac{\varphi}{2} = \frac{\sqrt{[1+x^2]}+1}{x}$;

$x\,\mathfrak{Cot}\frac{\varphi}{2} - 1 = \sqrt{[1+x^2]}$; $x^2\,\mathfrak{Cot}^2\frac{\varphi}{2} - 2x\,\mathfrak{Cot}\frac{\varphi}{2} = {}^2$; $x\left(\mathfrak{Cot}^2\frac{\varphi}{2} - 1\right) = 2\,\mathfrak{Cot}\frac{\varphi}{2}$;

$$\frac{x}{\mathfrak{Sin}^2\frac{\varphi}{2}} = 2\,\frac{\mathfrak{Cos}\frac{\varphi}{2}}{\mathfrak{Sin}\frac{\varphi}{2}}$$

$$x = 2\,\mathfrak{Cos}\frac{\varphi}{2}\,\mathfrak{Sin}\frac{\varphi}{2}; = 2\,\frac{e^{\frac{\varphi}{2}} + e^{-\frac{\varphi}{2}}}{2}\cdot\frac{e^{\frac{\varphi}{2}} - e^{-\frac{\varphi}{2}}}{2} = \frac{e^{\varphi} - e^{-\varphi}}{2} = \mathfrak{Sin}\,\varphi,$$

unde $\varphi = \mathfrak{Arc}\,(\mathfrak{Sin} = x)$; quo integrale reductum est ad

$$\int \frac{dx}{\sqrt{[1+x^1]}} = C + \mathfrak{Arc}\,(\mathfrak{Sin} = x)$$

Quarta formula integrationis exhibet nobis

4. $\int \frac{dx}{\sqrt{[1+x^2]}} = \int \frac{-2dz}{-1+z^2} = \int \frac{2dz}{1-z^2}\left(\text{cond: } z = \frac{\sqrt{[1+x^2]}-1}{x}\right)$

Est autem $\mathfrak{Cos}^2 u - \mathfrak{Sin}^2 u = 1$; $1 - \mathfrak{Tng}^2 u = \frac{1}{\mathfrak{Cos}^2 u}$. Faciamus $z = \mathfrak{Tng}\,u$; ergo $u = \mathfrak{Arc}\,(\mathfrak{Tng} = z)$; habebimus $dz = d\,\frac{\mathfrak{Sin}\,u}{\mathfrak{Cos}\,u} = \frac{du}{\mathfrak{Cos}^2 u}; = du\,(1-z^2)$;

ergo $\frac{dz}{1-z^2} = du$.

$$\int \frac{dz}{1-z^2} = C + \mathfrak{Arc}\,(\mathfrak{Tng} = z)$$

$$\int \frac{2dz}{1-z^2} = C + 2 \text{ Arc } (\text{Tng} = z)$$

unde concludimus

$$\int \frac{dx}{\sqrt{[1+x^2]}} = C + 2 \text{ Arc } \left(\text{Tng} = \frac{\sqrt{[1+x^2]} - 1}{x}\right)$$

Ad reductionem formandam faciamus $2 \text{ Arc } \left(\text{Tng} = \frac{\sqrt{[1+x^2-1}}{x}\right) = \varphi$; unde sequetur $\text{Tng } \frac{\varphi}{2} = \frac{\sqrt{[1+x^2]-1}}{x}$; $1 + x \text{ Tng } \frac{\varphi}{2} = \sqrt{[1+x^2]}$; $1 + 2 x \text{ Tng } \frac{\varphi}{2} + x^2 \text{ Tng}^2 \frac{\varphi}{2} = 1 + x^2$ $2 \text{ Tng } \frac{\varphi}{2} = x\left(1 - \text{Tng}^2 \frac{\varphi}{2}\right) = x \cdot \frac{1}{\text{Cos}^2 \frac{\varphi}{2}}$; $x = 2 \text{ Sin } \frac{\varphi}{2} \text{ Cos } \frac{\varphi}{2} = \text{Sin } \varphi$;

denique $\varphi = \text{Arc } (\text{Sin} = x)$; unde iterum

$$\int \frac{dx}{\sqrt{[1+x^2}} = C + \text{Arc } (\text{Sin} = x).$$

Restat formula quinta generalis integrandi, qua formula adhibita inducitur quantitas imaginaria, qua uti sine ulla dubitatione licet; cum autem in ejus reductione et transformatione aliquantum salebrosi lateat, totum calculum persequi haud alienum est. Jn integratione secundum normam quintam ad functionem

$$\int \frac{dx}{\sqrt{[1+x^2]}}$$

applicata facimus $\sqrt{[1+x^2]} = \sqrt{\left((1+x\sqrt{[-1]})\,(1-x\sqrt{[-1]})\right)}$; deinde introducimus aliam functionem id genus, ut sit $\sqrt{\left((1+x\sqrt{[-1]})\,(1-x\sqrt{[-1]})\right)} = (1+x\sqrt{[-1]})\,z$; quo prodibit $(1 + x\sqrt{[-1]})\,(1-x\sqrt{[-1]}) = (1+x\sqrt{[-1]})^2 z^2$; $1 - x\sqrt{[-1]} = (1+x\sqrt{[-1]})\,z^2 = z^2 + z^2 x\sqrt{[-1]}$; $1-z^2 = (1+z^2)\,x\sqrt{[-1]}$; $x\sqrt{[-1]} = \frac{1-z^2}{1+z^2}$;

$\sqrt{[-1]}\,dx = \frac{(1+z^2)\,(-2z) - (1-z^2)\,2z}{(1+z^2)^2}\,dz = \frac{-4z}{(1+z^2)^2}\,dz$; $dx = \frac{-4z}{(1+z^2)^2\sqrt{[-1]}}\,dz$;

$\sqrt{[1+x^2]} = (1+x\sqrt{[-1]})\,z = \left(1+\frac{1-z^2}{1+z^2}\right)z = \frac{2z}{1+z^2}$; unde conjungendo expressas functiones $\frac{dx}{\sqrt{[1+x^2]}} = \frac{-2dz}{(1+z^2)\sqrt{[-1]}}$; ergo

$$\int \frac{dx}{\sqrt{[1+x^2]}} = \int \frac{-2dz}{(1+z^2)\sqrt{[-1]}}; \left(\text{cond: } z = \sqrt{\frac{1-x\sqrt{[-1]}}{1+x\sqrt{[-1]}}}\right)$$

Est autem notissimus canon

$$\int \frac{dz}{1+z^2} = C + \text{arc } (\text{tng} = z);$$ unde deducimus

5. $\int \frac{dx}{\sqrt{[1+x^2]}} = C - \frac{2}{\sqrt{-1}} \text{ arc } \left(\text{tng} = \sqrt{\frac{1-x\sqrt{-1}}{1+x\sqrt{-1}}}\right).$

Eam formulam ut transformemus, facimus $2 \text{ arc } \left(\text{tng} = \sqrt{\frac{1-x\sqrt{-1}}{1+x\sqrt{-1}}}\right) = \varphi$;

unde deducimus $\text{tng } \frac{\varphi}{2} = \sqrt{\frac{1-x\sqrt{-1}}{1+x\sqrt{-1}}}$; $\text{tng}^2 \frac{\varphi}{2} = \frac{1-x\sqrt{-1}}{1+x\sqrt{-1}}$;

$\text{tng}^2 \frac{\varphi}{2} + x\sqrt{-1} \,.\, \text{tng}^2 \frac{\varphi}{2} = 1 - x\sqrt{-1}$; $x\sqrt{-1}\left(1 + \text{tng}^2 \frac{\varphi}{2}\right) = 1 - \text{tng}^2 \frac{\varphi}{2}$;
$\frac{x\sqrt{-1}}{\cos^2 \frac{\varphi}{2}} = \frac{\cos \varphi}{\cos^2 \frac{\varphi}{2}}$; $x \sqrt{-1} = \cos \varphi = \sin\left(\frac{\pi}{2} - \varphi\right)$; $\frac{\pi}{2} - \varphi$
$= \text{arc } (\sin = x\sqrt{-1})$, quare

$$\int \frac{dx}{\sqrt{[1+x^2}} = C - \frac{\pi}{2\sqrt{-1}} + \frac{1}{\sqrt{-1}} \text{ arc } (\sin = x\sqrt{-1});$$

$= C' + \frac{1}{\sqrt{-1}} \text{ arc } (\sin = x \sqrt{-1})$. Denuo facimus $\frac{1}{\sqrt{-1}} \text{ arc } (\sin = x\sqrt{-1})$
$= \psi$; unde sequitur, ut sit $\sin (\psi \sqrt{-1}) = x \sqrt{-1}$.

Transeundo ad formulas fundamentales est $x \sqrt{-1} = \frac{1}{2\sqrt{-1}}\left\{e^{-\psi} - e^{\psi}\right\}$;

$$x = -\tfrac{1}{2}\left\{e^{-\psi} - e^{\psi}\right\} = \tfrac{1}{2}\left\{e^{\psi} - e^{-\psi}\right\} = \mathfrak{Sin}\ \psi;\ \psi = \mathfrak{Arc}\ (\mathfrak{Sin} = x)$$

unde ad finem perducta transformatione efficitur, ut sit denique

$$\int \frac{dx}{\sqrt{[1+x^2}} = C + \mathfrak{Arc}\ (\mathfrak{Sin} = x).$$

IV.

Functio

$$\int \frac{dz}{\sqrt{[-1-x^2]}}$$

nullo modo ita integrari potest, ut unquam realis quantitas inde oriatur, quamcunque ex quinque propositis rationibus adhibebis, transformata igitur sejunctione unitatis imaginariae exhibebit

$$\int \frac{dx}{\sqrt{-1}\sqrt{[1+x^2]}} = C - \sqrt{-1}\ \mathfrak{Arc}\ (\mathfrak{Sin} = x)$$

eaque forma integrationis est ceteris praeferenda, cum statim generalis formulae $a+b\sqrt{-1}$ speciem prae se ferat. $\mathfrak{Arc}\ (\mathfrak{Sin} = x)$ est enim realis quantitas, quae facile calculo inveniri potest ideoque arcui cyclico praestat. Si enim ponis $\sqrt{-1}\ \mathfrak{Arc}\ (\mathfrak{Sin} = x) = \varphi$, deduces inde Sin $(\varphi\sqrt{-1}) = x$; $\dfrac{e^{\varphi\sqrt{-1}} - e^{-\varphi\sqrt{-1}}}{2} = \sqrt{-1}\sin\varphi = x$;

$$\sin\varphi = \frac{x}{\sqrt{-1}};\ \varphi = \text{arc}\,(\sin = -x\sqrt{-1})$$

$$\int\frac{dx}{\sqrt{[-1-x^2]}} = C - \text{arc}\,(\sin = -\sqrt{-1}).$$

Restat igitur tractanda functio

$$\int\frac{dx}{\sqrt{[x^2-1]}}$$

Ut autem in adhibenda prima et secunda formula generali calculi implicationem evites, in altera adhibe substitutionem $\sqrt{[x^2-1]} = -x + z$; et in altera $\sqrt{[x^2-1]} = x - z$; exinde deduces rebus hactenus tractatis congruentia, videlicet

1. $$\int\frac{dx}{\sqrt{[x^2-1]}} = C + \log\left\{x + \sqrt{[x^2-1]}\right\}$$

2. $$\int\frac{dx}{\sqrt{[x^2-1]}} = C - \log\left\{x - \sqrt{[x^2-1]}\right\}$$

Jn utraque formula quantitas constans eadem est, cum pro valore $x = 1$ logarithmi evanescant, quare sumto arcu φ ponimus

$$\varphi = \log\left\{x + \sqrt{[x^2-1]}\right\};\ -\varphi = \log\left\{x - \sqrt{[x^2-1]}\right\}$$

unde deducimus ex logarithmis ad numeros ascendendentes $e^{\varphi} = x + \sqrt{[x^2-1]}$

$e^{-\varphi} = x - \sqrt{[x^2-1]}$; quare addendo et subtrahendo $\dfrac{e^{\varphi} + e^{-\varphi}}{2} = x$

$\dfrac{e^{\varphi} - e^{-\varphi}}{2} = \sqrt{[x^2-1]}$; unde ponendum est $x =$ Cos φ; $\varphi =$ Arc (Cos $= x$);

$\sqrt{[x^2-1]} =$ Sin φ; $\varphi =$ Arc (Sin $= \sqrt{[x^2-1]}$, jamque habemus

$$\int\frac{dx}{\sqrt{[x^2-1]}} = C + \text{Arc}\,(\text{Cos} = x)$$

Arc (Cos $= x) = \log\,(x + \sqrt{[x^2-1]}) = -\log\,(x - \sqrt{[x^2-1]})$.

Jam si tertiam vel quartam formulam adhibere vis, tricae tibi parabuntur molestissimae, quippe qui intempestivo et praepostero modo quantitatem imaginariam inducas, quam deinde amovere multi laboris est. Etenim si in functione $\int\frac{dx}{\sqrt{[x^2-1]}}$ quantitas realis $x < 1$ est, functio integranda est imaginaria, quare habebis

$$\int\frac{dx}{\sqrt{[x^2-1]}} = \int\frac{dx}{\sqrt{[1-x^2]}\,\sqrt{-1}} = C - \frac{1}{\sqrt{-1}}\,\text{arc}\,(\cos = x)$$

Jam fac $-\frac{1}{\sqrt{-1}}$ arc (cos $= x) = \varphi$; et habebis arc (cos $= x) = -\varphi\sqrt{-1}$

ergo $\cos(-\varphi\sqrt{-1}) = x$; $\cos(-\varphi\sqrt{-1}) = \frac{e^{\varphi} + e^{-\varphi}}{2} = \mathfrak{Cos}\,\varphi$

$\varphi = \mathfrak{Arc}\,(\mathfrak{Cos} = x)$

$$\int \frac{dx}{\sqrt{[x^2-1]}} = C^1 + \mathfrak{Arc}\,(\mathfrak{Cos} = x).$$

Ne autem mireris, tibi arcum hyperbolicum realem evenisse, quamquam functio integranda sit imaginaria; unitas enim imaginaria tibi non periit, sed latet in Constanti, quoniam, uti facile invenies, est $C \pm 2\,m\,\pi\sqrt{-1} + \mathfrak{Arc}\,(\mathfrak{Cos} = x) = \mathfrak{Arc}\,(\mathfrak{Cos} = x] + C^1$ qua formula integratio rediit ad generalem typum $a + b\sqrt{-1}$.

Ut autem in casu $x > 1$ eodem reducatur integratio, ponimus $\sqrt{[x^2-1]} = x\,z + \sqrt{-1}$ quod aliquatenus praepostere factum est, cum x nunc sit quantitas realis. Jam persequendo calculo substitutionis est $\frac{x}{\sqrt{-1}} = \frac{2z}{1-z^2}$; $\frac{dx}{\sqrt{-1}} = \frac{2\,(1+z^2)}{(1-z^2)^2}\,dz$

$$\sqrt{[x^2-1]} = \sqrt{-1}\,\frac{1+z^2}{1-z^2}$$

$$\int \frac{dx}{\sqrt{[x^2-1]}} = \int \frac{2dz}{1-z^2}\ \left(\text{cond: } z = \frac{\sqrt{[x^2-1]} - \sqrt{-1}}{x}\right)$$

Quod integrale ut inveniatur, sume $\mathfrak{Tng}\,y = v$; $y = \mathfrak{Arc}\,(\mathfrak{Tng} = v)$; et habebis $d\,\mathfrak{Tng}\,y = dv = \frac{dy}{\mathfrak{Cos}^2 y}$; et cum ex $\mathfrak{Cos}^2 y - \mathfrak{Sin}^2 y = 1$ sequatur, ut sit

$1 - \mathfrak{Tng}^2 y = \frac{1}{\mathfrak{Cos}^2 y}$; habebimus $dv = dy\,(1 - \mathfrak{Tng}^2 y) = dy\,(1 - v^2)$; $dy = \frac{dv}{1-v^2}$

$$\int \frac{dv}{1-v^2} = C + y = C + \mathfrak{Arc}\,(\mathfrak{Tng} = v)$$

unde deducimus

$$\int \frac{2dz}{1-z^2} = C + 2\,\mathfrak{Arc}\,(\mathfrak{Tng} = z)$$

quare est:

$$\int \frac{dx}{\sqrt{[x^2-1]}} = C + 2\,\mathfrak{Arc}\left(\mathfrak{Tng} = \frac{\sqrt{[x^2-1]} - \sqrt{-1}}{x}\right]$$

Ut transformemus, facimus $\mathfrak{Arc}\left(\mathfrak{Tng} = \frac{\sqrt{[x^2-1]} - \sqrt{-1}}{x}\right) = \frac{\varphi}{2}$; quare est

$\mathfrak{Tng}\,\frac{\varphi}{2} = \frac{\sqrt{[x^2-1]} - \sqrt{-1}}{x}$; vel nominatorem et denominatorem unitate imaginaria multiplicando $\mathfrak{Tng}\,\frac{\varphi}{2} = \frac{\sqrt{[1-x^2]} + 1}{x\sqrt{-1}}$; $\sqrt{-1}\,\mathfrak{Tng}\,\frac{\varphi}{2} = \frac{1 + \sqrt{[1-x^2]}}{x}$

Est autem sin $\left(\frac{\varphi}{2}\sqrt{-1}\right) = \mathfrak{Sin}\,\frac{\varphi}{2}\,.\,\sqrt{-1}$; cos $\left(\frac{\varphi}{2}\sqrt{-1}\right) = \mathfrak{Cos}\,\frac{\varphi}{2}$

ergo $\sqrt{-1}\ \mathfrak{Tng}\,\frac{\varphi}{2} = \text{tng}\left(\frac{\varphi}{2}\sqrt{-1}\right)$; unde habemus tng $\left(\frac{\varphi}{2}\sqrt{-1}\right) = \frac{1+\sqrt{[1-x^2]}}{x}$ x; tng $\left(\frac{\varphi}{2}\sqrt{-1}\right) - 1 = \sqrt{[1-x^2]}$;

$$x^2\,\text{tng}^2\left(\frac{\varphi}{2}\sqrt{-1}\right) - 2\,x\,\text{tng}\left(\frac{\varphi}{2}\sqrt{-1}\right) = -\,x^2$$

$$x\left(\text{tng}^2\left(\frac{\varphi}{2}\sqrt{-1}\right) + 1\right) = 2\,\text{tng}\left(\frac{\varphi}{2}\sqrt{-1}\right)$$

$$\frac{x\left\{\sin^2\left(\frac{\varphi}{2}\sqrt{-1}\right) + \cos^2\left(\frac{\varphi}{2}\sqrt{-1}\right)\right\}}{\cos^2\left(\frac{\varphi}{2}\sqrt{-1}\right)} = 2\,\text{tng}\left(\frac{\varphi}{2}\sqrt{-1}\right)$$

Est $\sin^2\left(\frac{\varphi}{2}\sqrt{-1}\right) + \cos^2\left(\frac{\varphi}{2}\sqrt{-1}\right) = -\,\mathfrak{Sin}^2\,\frac{\varphi}{2} + \mathfrak{Cos}^2\,\frac{\varphi}{2} = 1$; ergo

$$\frac{x}{\cos^2\left[\frac{\varphi}{2}\sqrt{-1}\right]} = 2\,\text{tng}\left(\frac{\varphi}{2}\sqrt{-1}\right);\ x = 2\,\sin\left(\frac{\varphi}{2}\sqrt{-1}\right)\cos\left(\frac{\varphi}{2}\sqrt{-1}\right)$$

$= \sin(\varphi\sqrt{-1})$. Ut autem amoveatur unitas imaginaria, transeundum est ad cosinum.

Est autem $\cos\left(\frac{\pi}{2} + \varphi\sqrt{-1}\right) = \cos\frac{\pi}{2}\cos(\varphi\sqrt{-1}) - \sin\frac{\pi}{2}\sin(\varphi\sqrt{-1})$

$= -\sin(\varphi\sqrt{-1})$; ergo $-\,x = \cos\left(\frac{\pi}{2} + \varphi\sqrt{-1}\right)$; $x = \cos\left[-\frac{\pi}{2} - \varphi\sqrt{-1}\right]$;

$-\frac{\pi}{2} - \varphi\sqrt{-1} = \text{arc}\,(\cos = x)$; $-\frac{\pi}{2\sqrt{-1}} - \frac{1}{\sqrt{-1}}\,\text{arc}\,(\cos = x) = \varphi$

quare denique

$$\int\frac{dx}{\sqrt{[x^2-1]}} = C - \frac{\pi}{2\sqrt{-1}} - \frac{1}{\sqrt{-1}}\,\text{arc}\,(\cos = x)$$

$$= C - \frac{1}{\sqrt{-1}}\,\text{arc}\,(\cos = x)$$

$$= C + \mathfrak{Arc}\,(\mathfrak{Cos} = x)$$

Si denique quintam formulam generalem adhibemus, calculus est facillimus. Sumimus $\sqrt{[x^2-1]} = \sqrt{[(x+1)(x-1)]} = (x+1)\,z$; et invenimus

$$z = \sqrt{\frac{x-1}{x+1}};\ x = \frac{1+z^2}{1-z^2};\ dx = \frac{4z}{(1-z^2)^2}\,dz;\ \sqrt{[x^2-1]} = \frac{2z}{1-z^2};\ \text{unde est}$$

$$\int\frac{dx}{\sqrt{[x^2-1]}} = \int\frac{2dz}{1-z^2} = C + 2\,\mathfrak{Arc}\left(\mathfrak{Tng} = \sqrt{\frac{x-1}{x+1}}\right).$$ Jam ponas

$2\,\mathfrak{Arc}\left\{\mathfrak{Tng} = \sqrt{\frac{x-1}{x+1}}\right\} = \varphi$; et habebis $\mathfrak{Tng}\,\frac{\varphi}{2} = \sqrt{\frac{x-1}{x+1}}$

$\mathfrak{Tng}^2\,\frac{\varphi}{2} + x\,\mathfrak{Tng}^2\,\frac{\varphi}{2} = x - 1$; $\mathfrak{Tng}^2\,\frac{\varphi}{2} + 1 = x - x\,\mathfrak{Tng}^2\,\frac{\varphi}{2}$

$\mathfrak{Sin}^2\,\frac{\varphi}{2} + \mathfrak{Cos}^2\,\frac{\varphi}{2} = x$; $\mathfrak{Cos}\,\varphi = x$; unde revertisti ad integrale

$$\int\frac{dx}{\sqrt{[x^2-1]}} = C + \text{Arc}\,(\text{Cos} = x)$$

uti praevidimus.

V.

Hactenus quae exposita sunt, viam nobis sternunt ad integrationem functionum magis implicatarum. Jnitio potest loco unitatis semper alia quantitas constans poni, neque est difficile, eum casum reducere ad simpliciorem jam tractatum. Quae cum integrationes in theoria curvarum geometrica occurrant, operae pretium est, eas enumerare. Faciendo enim $y = \sqrt{\frac{c}{a}}\,x$, invenimus

$$\int\frac{dx}{\sqrt{(a-cx^2)}} = \int\frac{dy}{\sqrt{c}\sqrt{(1-y^2)}} = C + \frac{1}{\sqrt{c}}\,\text{arc}\,\left(\sin = x\sqrt{\frac{c}{a}}\right)$$

$$\int\frac{dx}{\sqrt{(a+cx^2)}} = \int\frac{dy}{\sqrt{c}\sqrt{(1+y^2)}} = C + \frac{1}{\sqrt{c}}\,\text{Arc}\,\left(\text{Sin} = x\sqrt{\frac{c}{a}}\right)$$

$$\int\frac{dx}{\sqrt{(cx^2-a)}} = \int\frac{dy}{\sqrt{c}\sqrt{(y^2-1)}} = C + \frac{1}{\sqrt{c}}\,\text{Arc}\,\left(\text{Cos} = x\sqrt{\frac{c}{a}}\right)$$

Ut autem statim ad casum generalem simpliciores omnes una formula continentem transeamus, nunc tractemus functionem

$$\int\frac{dx}{\sqrt{(a+bx-cx^2)}}$$

quae facile ita transformari potest, ut simpliciorem speciem $\int\frac{dy}{\sqrt{(1-y^2)}}$ prae se ferat, quo ad sinum cyclicum deducimur. Observemus, esse

$$\sqrt{(a+bx-cx^2)} = \sqrt{c}.\sqrt{\left(\frac{a}{c}+\frac{b}{c}-x^2\right)};\ -\left(x^2-\frac{b}{2c}\right)^2+\frac{b^2}{4c^2} =$$

$$-x^2+\frac{b}{c}\,x-\frac{b^2}{4c^2}+\frac{b^2}{4c^2} = \frac{b}{c}\,x - x^2;$$ et inveniemus

$$\int\frac{dx}{\sqrt{(a+bx-cx^2)}} = \int\frac{dx}{\sqrt{c}\sqrt{\left(\frac{a}{c}+\frac{b^2}{4c^2}-\left(x-\frac{b}{2c}\right)^2\right)}}$$

Jam fac $\frac{a}{c}+\frac{b^2}{4c^2} = m^2$; ergo $m = \sqrt{\left(\frac{a}{c}+\frac{b^2}{4c^2}\right)}$

$x - \frac{b}{2c} = y$; ergo $dx = dy$; et substituendo habebis

$$\int\frac{dx}{\sqrt{(a+bx-cx^2)}} = \int\frac{dy}{\sqrt{c}\sqrt{(m^2-y^2)}}$$

$$= C + \frac{1}{\sqrt{c}} \text{ arc } \left(\sin = \frac{y}{m}\right)$$

$$= C - \frac{1}{\sqrt{c}} \text{ arc } \left(\cos = \frac{y}{m}\right)$$

cum autem sit

$$\frac{y}{m} = \frac{x - \frac{b}{2c}}{\sqrt{\left(\frac{a}{c} + \frac{b^2}{4c^2}\right)}} = \frac{2c\ x - b}{\sqrt{(4ac + b^2)}} \text{ ; prodibit}$$

$$\int \frac{dx}{\sqrt{(a+bx-cx^2)}} = C + \frac{1}{\sqrt{c}} \text{ arc } \left(\sin = \frac{2cx-b}{\sqrt{(4ac+b^2)}}\right)$$

Quod integrale inventum comparemus cum iis quantitatibus, quas adhibitis quinque methodis initio explicatis inveniemus. Etenim si utimur prima et altera formula generali, prodibit

1. $$\int \frac{dx}{\sqrt{(a+bx-cx^2)}} = \int \frac{2dz}{b+2z\sqrt{-c}} = C^1 + \frac{1}{\sqrt{-c}} \log (2z \sqrt{-c} + b)$$

$$(\text{cond: } z = \sqrt{(a+bx-cx^2)} + x\sqrt{-c})$$

$$= C^1 + \frac{1}{\sqrt{-1}\sqrt{c}} \log \left\{ 2\sqrt{-c}\sqrt{(a+bx - cx^2)} - (2cx - b) \right\}$$

2. $$\int \frac{dx}{\sqrt{(a+bx-cx^2)}} = \int \frac{-2dz}{2z\sqrt{-c}-b} = C^1 - \frac{1}{\sqrt{-c}} \log \left\{ 2z\sqrt{-c}-b \right\}$$

$$(\text{cond: } z = \sqrt{(a+bx-cx^2)} - x\sqrt{-c})$$

$$= C^1 - \frac{1}{\sqrt{-1}\sqrt{c}} \log \left\{ 2\sqrt{-c}\sqrt{(a+bx-cx^2)} + (2\ cx-b) \right\}$$

Jntroducendo autem quantitates jam nobis notas $m = \sqrt{\left(\frac{a}{c} + \frac{b^2}{4c^2}\right)}$; $y = x - \frac{b}{2c}$, illos logarithmos permutabimus in hos:

$$C^1 + \frac{1}{\sqrt{-1}\sqrt{c}} \log \left\{ 2\ c\sqrt{(m^2-y^2)}\sqrt{-1} - 2\ cy \right\}$$

$$C^1 - \frac{1}{\sqrt{-1}\sqrt{c}} \log \left\{ 2\ c\sqrt{(m^2-y^2)}\sqrt{-1} + 2\ cy \right\}$$

Quodsi in utroque multiplicatorem $2cm\sqrt{-1}$ sejungimus, et re confecta quantitatem invariabilem $\frac{1}{\sqrt{-1}\sqrt{c}} \log (2cm\sqrt{-1})$ cum C^1 conjungimus, quod in integrationibus licitum esse constat, prodibit

$$\int \frac{dx}{\sqrt{(a+bx-cx^2)}} = C^1 + \frac{1}{\sqrt{-1-c}} \log \left\{ \sqrt{\left(1- \frac{y^2}{m^2}\right)} + \frac{y}{m}\sqrt{-1} \right\}$$

$$\int\frac{dx}{\sqrt{(a+bx-cx^2)}} = C^1 - \frac{1}{\sqrt{-1}\sqrt{c}}\log\left\{\sqrt{(1-\frac{y^2}{m^2})} - \frac{y}{m}\sqrt{-1}\right\}$$

Jam videmus. pro valore $y = 0$ utrumque logarithmum evanescere, ideoque in his formulis quantitatem constantem C^1 eandem esse, quare sumto arcu $\frac{\varphi}{\sqrt{c}}$ habemus jam

$$\varphi = \frac{1}{\sqrt{-1}}\log\left\{\sqrt{(1-\frac{y^2}{m^2})} + \frac{y}{m}\sqrt{-1}\right\}$$

$$-\varphi = \frac{1}{\sqrt{-1}}\log\left\{\sqrt{(1-\frac{y^2}{m^2})} - \frac{y}{m}\sqrt{-1}\right\}$$

unde transeundo ex logarithmis ad numeros congruentes, tum addendo et subtrahendo

est $\sqrt{(1-\frac{y^2}{m^2})} = \frac{e^{\varphi\sqrt{-1}} + e^{-\varphi\sqrt{-1}}}{2} = \cos\varphi$

$$\frac{y}{m} = \frac{e^{\varphi\sqrt{-1}} - e^{-\varphi\sqrt{-1}}}{2\sqrt{-1}} = \sin\varphi$$

quare ex logarithmis ad arcum $\frac{\varphi}{\sqrt{c}}$ revertimus et iterum invenimus

$$\int\frac{dx}{\sqrt{(a+bx-cx^2)}} = C + \frac{\varphi}{\sqrt{c}} = C + \frac{1}{\sqrt{c}}\text{ arc }(\sin = \frac{y}{m})$$

$$= C + \frac{1}{\sqrt{c}}\text{ arc }(\sin = \frac{2\,cx-b}{\sqrt{(4ac+b^2)}})$$

Sequitur tertia et quarta integrationis formula generalis, qua adhibita invenimus

3. $\int\frac{dx}{\sqrt{(a+bx-cx^2)}} = \int\frac{-2dz}{c+z^2}$ (cond: $z = \frac{\sqrt{(a+bx-cx^2)}+\sqrt{a}}{x}$)

4. $\int\frac{dx}{\sqrt{(a+bx-cx^2)}} = \int\frac{-2dz}{c+z^2}$ (cond: $z = \frac{\sqrt{(a+bx-cx^2)}-\sqrt{a}}{x}$)

Est autem notissimus canon:

$\int\frac{-2dz}{c+z^2} = C - \frac{2}{\sqrt{c}}\text{ arc }(\text{tng} = \frac{z}{\sqrt{c}})$; quare deducimus formulas

$$\int\frac{dx}{\sqrt{(a+bx-cx^2)}} = C - \frac{2}{\sqrt{c}}\text{ arc }(\text{tng} = \frac{\sqrt{(a+bx-cx^2)}+\sqrt{a}}{x\sqrt{c}})$$

$$\int\frac{dx}{\sqrt{(a+bx-cx^2)}} = C - \frac{2}{\sqrt{c}}\text{ arc }(\text{tng} = \frac{\sqrt{(a+bx-cx^2)}-\sqrt{a}}{x\sqrt{c}})$$

Cum autem hos arcus tangentium cyclicarum ad arcum sinus reducere volumus, irretimur ridicule prolixo calculo, quare introducimus denuo valores

$m^2 = \frac{a}{c} + \frac{b^2}{4c^2}$; $y = x - \frac{b}{2c}$ unde transformatione absoluta et tertia quartaque integrationis formula adhibita prodibit

$$\int \frac{dy}{\sqrt{c}\sqrt{(m^2-y^2)}} = C - \frac{2}{\sqrt{c}} \text{ arc } \left(\text{tng} = \frac{\sqrt{(m^2-y^2)}+m}{y}\right)$$

$$= C - \frac{2}{\sqrt{c}} \text{ arc } \left(\text{tng} = \frac{\sqrt{(m^2-y^2)}-m}{y}\right)$$

Jam autem transitus ad arcum sinus facillimus et supra monstratus est. Si enim facimus $2 \text{ arc } \left(\text{tng} = \frac{\sqrt{(m^2-y^2)}+m}{y}\right) = \varphi$, inde deducimus

$$m + y \text{ tng } \tfrac{\varphi}{2} = \sqrt{(m^2-y^2)};\ y + m \sin \varphi = 0$$

$$\frac{y}{m} = - \sin \varphi = \sin(-\varphi);\ -\varphi = + \text{ arc } \left(\sin = \frac{y}{m}\right)$$

et si facimus $2 \text{ arc } \left(\text{tng} = \frac{\sqrt{(m^2-y^2)}-m}{y}\right) = \varphi$, inde habemus

$$-m + y \text{ tng } \tfrac{\varphi}{2} = \sqrt{(m^2-y^2)};\ y = m \sin \varphi;\ \frac{y}{m} = \sin \varphi = \sin(\pi - \varphi)$$

$$\pi - \varphi = + \text{ arc } \left(\sin = \frac{y}{m}\right);\ -\varphi = -\pi + \text{ arc } \left(\sin = \frac{y}{m}\right)$$

Denique quinta formula integrationis generali utentes invenimus

5. $$\int \frac{dx}{\sqrt{(a+bx-cx^2)}} = \int \frac{-2dz}{\sqrt{c}(1+z^2)} = C - \frac{2}{\sqrt{c}} \text{ arc } (\text{tng} = z)$$

(cond: $z = \sqrt{\frac{r-x}{x-R}}$ ubi r et R sunt radices aequationis

$$x^2 - \frac{b}{c}x - \frac{a}{c} = 0;\ \text{nempe } x = \frac{b \pm \sqrt{(4ac+b^2)}}{2c}$$

quare prodit substitutione absoluta

$$\int \frac{dx}{\sqrt{(a+bx-cx^2)}} = C - \frac{2}{\sqrt{c}} \text{ arc } \left(\text{tng} = \frac{-2cx+b+\sqrt{(4ac+b^2)}}{2cx-b+\sqrt{(4ac+b^2)}}\right)$$

qui ut arcus tangentis ad arcum sinus reducatur, nominatorem et denominatorem fractionis divide quantitate $2c = \sqrt{[4c^2]}$ et habebis

$$C - \frac{2}{\sqrt{c}} \text{ arc } \left(\text{tng} = \sqrt{\frac{-y+m}{y+m}}\right);\ \text{jam fac } 2 \text{ arc } \left(\text{tng} = \sqrt{\frac{-y+m}{y+m}}\right)$$

$$= \varphi, \text{ et habebis tng } \tfrac{\varphi}{2} = \sqrt{\frac{-y+m}{y+m}};\ y \text{ tng}^2 \tfrac{\varphi}{2} + m \text{ tng}^2 \tfrac{\varphi}{2} = -y + m$$

$$\frac{y}{\cos^2 \frac{\varphi}{2}} = m\left(1 - \text{tng}^2 \tfrac{\varphi}{2}\right);\ y = m \cos \varphi = m \sin\left(\tfrac{\pi}{2} - \varphi\right)$$

$-\varphi = -\frac{\pi}{2} + \text{ arc } \left(\sin = \frac{y}{m}\right)$; unde iterum reductio absoluta est ad arcum sinus cyclici:

$$\int \frac{dx}{\sqrt{(a+bx-cx^2)}} = C + \text{arc}\left(\sin = \frac{y}{m}\right)$$

Patet, omnium integrationis formularum generalium commodissimam fere esse quintam vel tum, quum quantitates imaginariae oboriantur. Jam transibimus ad arcus hyperbolicos.

VI.

Jn integratione functionis

$$\int \frac{dz}{\sqrt{(a+bx-cx^2)}}$$

permutabis initio radicem in $\sqrt{c}\sqrt{\left(\frac{a}{c} + \frac{bx}{c} + x^2\right)}$. Est autem $x^2 + \frac{bx}{c} = -\frac{b^2}{4c^2} + \left(x + \frac{b}{2c}\right)^2$; et sumendo $u^2 = \frac{a}{c} - \frac{b^2}{4c^2} = \frac{4ac-b^2}{4c^2}$;

$t = x + \frac{b}{2c} = \frac{2cx+b}{2c}$; transmutatio absoluta est in hanc formulam:

$$\int \frac{dx}{\sqrt{(a+bx+cx^2)}} = \int \frac{dt}{\sqrt{c}\sqrt{(u^2+t^2)}},$$

ex ea autem concludimus:

$$\int \frac{dx}{\sqrt{(a+bx+cx^2)}} = C + \frac{1}{\sqrt{c}}\ \text{Arc}\left[\text{Sin} = \frac{t}{u}\right]$$

$$= C + \frac{1}{\sqrt{c}}\ \text{Arc}\left[\text{Sin} = \frac{2cx+b}{\sqrt{(4ac-b^2)}}\right]$$

ad quod integrale ceterae formulae reduci possunt.

Si enim adhibemus primam et alteram methodum integrationis, invenimus

$$\int \frac{dx}{\sqrt{(a+bx+cx^2)}} = \int \frac{2dz}{2z\sqrt{c}+b} \quad (\text{cond}: z = \sqrt{(a+bx+cx^2)}+x\sqrt{c})$$

$$\int \frac{dx}{\sqrt{(a+bx+cx^2)}} = \int \frac{-2dz}{2z\sqrt{c}-b} \quad (\text{cond}: z = \sqrt{(a+bx+cx^2)} - x\sqrt{c})$$

unde efficitur, ut sit

1. $$\int \frac{dx}{\sqrt{(a+bx+cx^2)}} = C + \frac{1}{\sqrt{c}} \log\left(2\sqrt{c}\sqrt{(a+bx+cx^2)}+2cx+b\right)$$

2. $$\int \frac{dx}{\sqrt{(a+bx+cx^2)}} = C - \frac{1}{\sqrt{c}} \log\left(2\sqrt{c}\sqrt{(a+bx+cx^2)}-2cx-b\right)$$

Jntroducimus quantitates $\mu^2 = \frac{a}{c} - \frac{b^2}{4c^2}$, $t = x + \frac{b}{2c}$, quo logarithmicae functiones transformantur in

$$C + \frac{1}{\sqrt{c}} \log (2c\sqrt{(\mu^2+t^2)}+2ct)$$

$$C - \frac{1}{\sqrt{c}} \log (2c\sqrt{(\mu^2+t^2)}-2ct)$$

sive, sejuncto in utraque factore $2c\mu$, deinde invariabili quantitate $\frac{1}{\sqrt{c}} \log (2c\mu)$ cum Constanti conjuncta,

$$C + \frac{1}{\sqrt{c}} \log \left\{ \sqrt{[1+\frac{t^2}{\mu^2}]} + \frac{t}{m} \right\}$$

$$C - \frac{1}{\sqrt{c}} \log \left\{ \sqrt{[1+\frac{t^2}{\mu^2}]} - \frac{t}{m} \right\}$$

Jam vides, in casu $t = 0$ logarithmos evanescere, unde eandem constantem in utroque esse, et propterea licet utrumque logarithmum aequalem arcui $\frac{\varphi}{\sqrt{c}}$ ponere, ex qua re transeundo ad numeros statim efficitur, ut sit

$$\varphi = \log \left\{ \sqrt{[1+\frac{t^2}{\mu^2}]} + \frac{t}{\mu} \right\}; \; -\varphi = \log \left\{ \sqrt{(1+\frac{t^2}{\mu^2})} + \frac{t}{\mu} \right\}$$

$$\frac{t}{\mu} = \frac{e^{\varphi} - e^{-\varphi}}{2} = \mathrm{Sin}\,\varphi; \; \sqrt{[1+\frac{t^2}{\mu^2}]} = \frac{e^{\varphi} + e^{-\varphi}}{2} = \mathrm{Cos}\,\varphi.$$

Tertia et quarta integrationis formula generali adhibita invenimus

$$\int \frac{dx}{\sqrt{(a+bx+cx^2)}} = \int \frac{-2dz}{z^2-C} \quad \left(\text{cond: } z = \frac{\sqrt{(a+bx+cx^2)}+\sqrt{a}}{x}\right]$$

$$\int \frac{dx}{\sqrt{(a+bx+cx^2)}} = \int \frac{-2dz}{z^2-C} \quad \left(\text{cond: } z = \frac{\sqrt{(a+bx+cx^2)}-\sqrt{a}}{x}\right.$$

quare est

$$3. \quad \int \frac{dx}{\sqrt{(a+bx+cx^2)}} = C + \frac{2}{\sqrt{c}} \mathrm{Arc}\left(\mathrm{Tng} = \frac{\sqrt{(a+bx+cx^2)}+\sqrt{a}}{x\sqrt{c}}\right]$$

$$4. \quad \int \frac{dx}{\sqrt{(a+bx+cx^2)}} = C + \frac{2}{\sqrt{c}} \mathrm{Arc}\left(\mathrm{Tng} = \frac{\sqrt{(a+bx+cx^2)}-\sqrt{a}}{x\sqrt{c}}\right]$$

sive, valores $\mu^2 = \frac{a}{c} - \frac{b^2}{4c^2}$; $t = x + \frac{b}{2c}$ introducendo

$$\int \frac{dt}{\sqrt{c}\sqrt{(\mu^2+t^2)}} = C + \frac{2}{\sqrt{c}} \mathrm{Arc}\left(\mathrm{Tng} = \frac{\sqrt{(\mu^2+t^2)}+\mu}{t}\right)$$

$$\int \frac{dt}{\sqrt{c}\sqrt{(\mu^2+t^2)}} = C + \frac{2}{\sqrt{c}} \mathrm{Arc}\left(\mathrm{Tng} = \frac{\sqrt{(\mu^2+t^2)}-\mu}{t}\right)$$

quos tangentium hyperbolicarum arcus facile in arcus sinuum transformabis, quod, ne repetamus monstrata, in quarto casu calculo subjiciemus.

Si enim sumis 2 Arc $\left(\mathrm{Tng} = \frac{\sqrt{(\mu^2+t^2)}-\mu}{t}\right) = \varphi$; habebis

$$\mathrm{Tng}\,\frac{\varphi}{2} = \frac{\sqrt{(\mu^2+t^2)}-\mu}{t};\quad \mu + t\,\mathrm{Tng}\,\frac{\varphi}{2} = \sqrt{(\mu^2+t^2)}$$

$$2\,\mu t\,\mathrm{Tng}\,\frac{\varphi}{2} = t^2\left(1-\mathrm{Tng}^2\,\frac{\varphi}{2}\right);\quad 2\,\mu\,\mathrm{Sin}\,\frac{\varphi}{2}\,\mathrm{Cos}\,\frac{\varphi}{2} = t;\quad t = \mu\,\mathrm{Sin}\,\varphi$$

$\varphi = \mathrm{Arc}\left(\mathrm{Sin} = \frac{t}{\mu}\right)$; quo reductio absoluta est.

Adhibeamus quintam denique integrationis formulam, verum iterum introducamus functiones t et μ, unde transformatio deducitur

$$\int \frac{dt}{\sqrt{c}\sqrt{(\mu^2+t^2)}}$$

et sumamus $\sqrt{(\mu^2+t^2)} = \sqrt{[(\mu+t\sqrt{-1})(\mu-t\sqrt{-1})]}; = (\mu+t\sqrt{-1})z$

Jnde convincietur esse $\mu - t\sqrt{-1} = (\mu+t\sqrt{-1})z^2$

$$\frac{1-z^2}{1+z^2} = \frac{t}{\mu}\sqrt{-1};\ \frac{dt}{\mu} = \frac{-4zdz}{(1+z^2)^2\sqrt{-1}};\ \sqrt{(\mu^2+t^2)} = \frac{2\mu z}{1+z^2}$$

Ex his formulis componitur integrale

$$\int \frac{dt}{\sqrt{c}\sqrt{(\mu^2+t^2)}} = \int \frac{-2du}{\sqrt{c}\sqrt{(1+z^2)}\sqrt{-1}}$$

$$= C - \frac{2}{\sqrt{-1}\sqrt{c}}\ \mathrm{arc}\left(\mathrm{tng} = \frac{\mu-t\sqrt{-1}}{\mu+t\sqrt{-1}}\right)$$

Secundum normas jam supra expositas talis arcus imaginarius facile transformatur in arcum realem sinus hyperbolici, id quod consulto omittimus. Jam via strata est, qua ad integralia magis complicata progredi liceat.

Not. ( ) est loco signi integrationis, quod typothetam deficiebat.