Unciarum theoriae

partem secundam

scripsit

Piegsa.

Quum ante hos octo annos mihi contigisset, ut priorem unciarum theoriae partem in lucem protulissem, facile poterat afferri spes, fore ut, occasione oblata, alteram illius operis partem ederem; sed exiguum, quod mihi concessum est in hoc programmate spatium non permittit, ut plane ad finem perducatur inceptum; qua re sequitur tantum particula.

§. 32.

Si potestatem $(m + n)^s$ evolvamus in seriem, ubi m, n, s numeri positivi sint, tum eum terminum maximum esse contendimus, in quo exponens literae m et literae n ita sunt inter se, ut hae ipsae quantitates, aut ubi eorum (exponentium) ratio rationi m : n proxime accedit.

Terminus enim $(r + 1)^{tus}$ seriei, in quam potestas $(m + n)^s$ evolvi potest, per literam M; terminus

q^{tus} illum $(r + 1)^{tum}$ insequens et terminus
q^{tus} illum $(n + 1)^{tum}$ antecedens resp. per

$$\overset{-q}{M} \quad \text{et} \quad \overset{+q}{M}$$

designetur. Est igitur

$$M = \frac{s\,(s-1)\,.\,.\,.\,(s-r+1)}{1\,.\,2\,.\,.\,.\,.\,r}\,.\,m^{s-r}\,n^{r}$$

$$\overset{+q}{M} = \frac{s\,(s-1)\,.\,.\,.\,(s-r-q+1)}{1.\;2.\,.\,.\,.\,(r+q)}\,.\,m^{s-r-q}\,n^{r+q}$$

$$\overset{-q}{M} = \frac{s\,(s-1)\,.\,.\,.\,(s-r+q+1)}{1\,.\,2\,.\,.\,.\,(r-q)}\,.\,m^{s-r+q}\,n^{r-q}$$

ergo

$$\frac{M}{\overset{+q}{M}} = \frac{(r+q)\,(r+q-1)\,.\,.\,.\,(r+1)}{(s-r)\,(s-r-1)\,.\,.\,.\,(s-r-q+1)}\,.\left(\frac{m}{n}\right)^q$$

$$\frac{M}{\overset{-q}{M}} = \frac{(s-r+q)\,(s-r+q-1)\,.\,.\,.\,(s-r+1)}{r\,(r-1)\,(r-2)\,.\,.\,.\,(r-q+1)}\,.\left(\frac{n}{m}\right)^q$$

Jam si aut accurate aut approximative

$$s - r : r = m : n$$

est, tum aut accurate aut approximative erit

$$\frac{M_{+q}}{M} = \frac{(r+q)}{(s-r)} \frac{(r+q-1)}{(s-r-1)} \cdots \frac{(r-1)}{(s-r-q+1)} \cdot \left\{\frac{s-r}{r}\right\}^{q}$$

$$= \frac{\{(s-r)\,r + (s-r)\,q\}}{\{(s-r)\}} \frac{}{\{(s-r)\,r - r\}} \frac{\{(s-r)\,r + (s-r)\,(q-1)\}}{\{(s-r)\,r - 2r\}} \frac{\cdots}{\cdots} \frac{\{(s-r)\,r + (s-r).1\}}{\{(s-r)\,r - (q-1)\,r\}};$$

$$\frac{M_{-q}}{M} = \frac{\{(s-r)\,r + q\,r\}}{\{(s-r)\,r\}} \frac{\{(s-r)\,r + (q-1)\,r\}}{\{(s-r)\,r - r\}} \frac{\cdots}{\cdots} \frac{\{(s-r)\,r + r\}}{\{(s-r)\,r - (q-1)\,r\}}.$$

Quia $s - r$ et q semper positivi numeri esse debent, theorema secundum eam ipsam cum in §. 25. adhibitam concludendi rationem (comparando factores denominatoris cum factoribus numeratoris) verum esse apparet.

§. 33.

Stirlingius in libello, qui inscriptus est: **Methodus differentialis pag. 119** ss. quatuor docet series, quibus auxiliantibus uncia media magnarum potestatum erui possit. Theoremata ita audiunt:

A) Si index dignitatis sit numerus par appelletur 2n, vel si impar vocetur 2 n + 1; eritque ut uncia media ad summam omnium (unciarum) ejusdem dignitatis, ita unitas ad mediam proportionalem inter semicircumferentiam circuli ad alterutram serierum sequentium.

$$1^{0})\quad 2n + \frac{1^2 \cdot 2n}{2(2n+2)} + \frac{1^2 \cdot 3^2 \cdot 2n}{2.4(2n+2)(2n+4)} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 2n}{2.4.6(2n+2)(2n+4)(2n+6)} + \cdots$$

$$2^{0})\quad 2n + 1 - \frac{1^2 (2n+1)}{2(2n-3)} + \frac{1^2 \cdot 3^2 (2n+1)}{2.4(2n-3)(2n-5)} - \frac{1^2 \cdot 3^2 \cdot 5^2 (2n+1)}{2.4.6(2n-3)(2n-5)(2n-7)} + \cdots$$

B) Est summa omnium unciarum ad mediam ut unitas ad mediam proportionalem inter reciprocam semicircumferentiam circuli ad alterutram serierum insequentium:

$$3_{0})\quad \frac{1}{2n+1} + \frac{1}{2(2n+1)(2n+3)} + \frac{1^2 \cdot 3^2}{2.4(2n+1)(2n+3)(2n+5)} + \cdots$$

$$4_0) \quad \frac{1}{2n} - \frac{1}{2.\,2n(2n-2)} + \frac{1.\;3}{2.\,4.\,2n(2n-2)(2n-4)} - \ldots$$

Si summam unciarum omnium potestatis $(2n)^{tae}$ signo Σ_{2n}, et series quatuor Stirlingii resp. signis s, s_1 s_2 s_3 scribamus, theoremata illa hoc modo brevissime exprimi possunt:

$$(2n)_n : \Sigma_{2n} = 1 : \sqrt{\frac{\pi}{2} \cdot s} = 1 : \sqrt{\frac{\pi}{2} \cdot s_1}$$

$$\Sigma_{2n} : (2n)_n = 1 : \sqrt{\frac{2}{\pi} \cdot s_2} = 1 : \sqrt{\frac{2}{\pi} \cdot s_3}$$

Ac jam seriem s_3 veram non esse, aqua est limpidius, quoniam denominatoris factor semper = o esse, ergo valor ejusdem termini infinitus fieri debet, id quod sequentia exempla pro $n = 1$, et $n = 2$ adhibita explicabunt.

Pro $n = 1$ est $2n = 2$; $\Sigma_{2n} = 1 + 1 = 2$; $(2n)_n = 1$.

Ergo est

$$\Sigma_{2n} : (2n)_n = 1 : \sqrt{\frac{2}{\pi} \cdot s_3}$$

$$= 2 : 1 = 1 : \sqrt{\left\{\frac{2}{\pi}\left(\frac{1}{2} - \frac{1}{2.\,2.\,1\,(2.\,1-2)} + \ldots\right)\right\}}$$

$$2 : 1 = 1 : \sqrt{\left\{\frac{2}{\pi}\left(\frac{1}{2} - \infty\right)\right\}}$$

Pro $n = 2$ est $2n = 4$; $\Sigma_{2n} = 16$, $(2n)_{2n} = 4_2 = 6$.

ergo erit

$$\Sigma_{2n} : (2n)_n = 1 : \sqrt{\frac{2}{\pi}\, s_3}$$

$$= 16 : 6 = 1 : \sqrt{\left\{\frac{2}{\pi}\left(\frac{1}{2.2} - \frac{1}{2.2.2(2.2-2)} + \frac{1\;.\;3_2}{2.4.2.2(2.2-4)} - \ldots\right\}}$$

$$= 8 : 3 = 1 : \sqrt{\left\{\frac{2}{\pi}\left(\frac{1}{4} - \frac{1}{16} + \infty\right)\right\}}$$

Item demonstari potest, seriem s_1 esse **divergentem**, ad id, quod sibi Stirlingius proposuit computandum ineptam, quod nostrum judicium ita probatur verum esse.

Est

$$\frac{s_1}{2n+1} = 1 - \frac{1}{2(2n-3)} + \frac{(1.\;3)^2}{2.4(2n-3)(2n-5)} \div \frac{(1.\;3\;.\;5)^2}{2.4.6(2n-3)(2n-5)(2n7)} + \ldots$$

$$\dots + (-1)^{\alpha} \frac{\{1 . 3 . 5 \dots (2\alpha - 1)\}^2}{2.4.6 \dots 2\alpha (2n-3)(2n-5) \dots (2n-2\alpha-1)} + \dots$$

Faciendo $\alpha = n$ erit

$$\frac{s_1}{2n+1} = 1 - \frac{1}{2(2n-3)} + \frac{(1.\ 3)^2}{2.4(2n-3)(2n-5)} - \frac{(1.\ 3.\ 5)^2}{2.4.6(2n-3)(2n-5)(2n-7)} + \dots$$

$$\dots + (-1)^{n} \frac{\{1.\ 3.\ 5 \dots (2n-1)\}^2}{2.4.6 \dots 2n (2n-3)(2n-5) \dots 5.3.1(-1)} + \dots$$

Summam terminorum n primorum per S_n et terminum n_{tum} per t_n significando, erit

$$\frac{s_1}{2n+1} = S_n + (-1)^{n+1} t_n \cdot \frac{(2n+1)^2}{(2n+2)(-3)} + (-1)^{n+2} t_n \cdot \frac{\{(2n+1)(2n+3)\}^2}{(2n+2)(2n+4)(-3)(-5)} + \dots$$

$$= S_n + (-1)^{n} t_n \Big\{ \frac{(2n+1)^2}{3(2n+2)} + \frac{\{(2n+1)(2n+3)\}^2}{3.5(2n+2)(2n+4)} + \dots$$

$$\dots + \frac{\{(2n+1)(2n+3) \dots (2n+\beta-1)\}^2}{3.5 \dots (2\beta+1)(2n+2) \dots (2n+2\beta)} + \dots \Big\}$$

$$= S_n + (-1)^n \cdot t_n \cdot \varphi$$

Terminus generalis seriei φ sit T, atqe erit

$$T = \frac{(2n+1)^2 (2n+3)^2 \dots (2n+2\beta-1)^2}{3(2n+2)5(2n+4) \dots (2\beta+1)(2n+2\beta)}$$

Jam vero, si $n > 2$ est, pro quovis valore positivo literae β tributo est

$$(2n+1)^2 > 3(2n+2)$$
$$(2n+3)^2 > 5(2n+4)$$
$$\dots \dots$$
$$(2n+2\beta-1)^2 > (2\beta+1)(2n-2\beta)$$

ergo etiam esse debet

$$(2n+1)^2 (2n+3)^2 \dots (2n+2\beta-1)^2 > 3(2n+2)5(2n+4) \dots (2\beta+1)(2n+2\beta)$$

Itaque numerator semper excedit denominatorem, ergo unusquisque terminus seriei φ major est unitate. Ac numerus terminorum est infinitus, ergo etiam $\varphi = \infty$ erit. Dei n series φ multiplicata est in t_n. Quodsi igitur demonstrare possimus, esse $t_n \gtrless 0$, tota series $\frac{s_1}{2n+1}$ summa gauderet infinita.

Ac jam erat

$$t_n = \frac{\{1.\ 3.\ 5.\ 7.\ \ldots\ (2n-1)\}^2}{2.\ 4.\ 6.\ \ldots\ 2n(2n-3)\ (2n-5)\ \ldots\ 3.\ 1(-1).}$$

$$= \frac{1^2 .\ 3^2 .\ 5^2 .\ 7^2\ \ldots\ (2n-1)^2}{(-1).\ 2.\ 1.\ 4.\ 3.\ 6.\ 5.\ 8\ \ldots\ 2n\,(2n-3)}$$

et quum praeterea sit

$$1^2 > (-1).\ 2$$
$$3^2 > 1.\ 4$$
$$5^2 > 3.\ 6$$
$$7^2 > 5.\ 8$$
$$.\ \ .\ \ .\ \ .$$
$$(2n-1) > (2n-3)\ 2n$$

pro quovis valore literae $n > o$ adscripto, ergo sequitur, numeratorem valoris t_n majorem esse denominatore, ergo valorem t_n ipsius majorem esse unitate, unde colligimus esse

$$\frac{s_1}{2n+1} = \infty\ ; \quad S_1 = \infty$$

Itaque tota series s_1 divergit, ergo eam calculo inservire non posse patet.

His de causis factum est, quod hoc loco primam modo et tertiam seriem accuratius explicare nobiscum constituerimus. Sed antequam eas veras aut falsas esse doceamus, observare javat quantitatem Σ_{2n} facile ad calculum posse vocari. Est enim

$$(1+x)^\alpha = 1 + \alpha_1\, x + \alpha_2\, x^2 + \alpha_3\, x^3 + \ldots\ \alpha_1\, x^{n-1} + 1.\ x^\alpha$$

Ponendo $x = 1$ erit

$$2^\alpha = 1 + \alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_1 + 1$$

id est: 2_α aequale est summae omnium unciarum. Itaque est

$$\Sigma_{2n} = 2^{2n}$$

Formulae modo dictae a nobis nonnisi pro exponentibus paribus expressae sunt; sed valent illae etiam pro exponentibus imparibus, qui sint unitate minores. Est enim

$$(2n)_n : 2^{2n} = (2n-1)_n : 2^{2n-1}$$

Quod si pro vero accipimus, erit quoque

$$(2n)_n : 2 = (2n-1)_n : 1$$

sive

$$2\,(2n-1)_n = (2n)_n$$

id est

$$\frac{2\,(2n-1)\,(2n-2)\,\ldots\,(n+1)\,n}{1.\,2\,\ldots\,n} = \frac{2n\,(2n-1)\,\ldots\,(n+1)}{1.\,2.\,3\,\ldots\,n}$$

quae aequatio procul dubio identica vocari potest. Hanc ob rem, ut commodius agere possimus, licebet nos semper nonnisi uncias medias exponentium parium tractare.

Jam exponentes proportionum, quas aut veras aut falsas esse demonstraturi sumus, pro casu $(2n)^{to}$ resp. per T et T_1; pro casu $2\,(n+1)^{to}$ resp. per T^1 et T_1^1 notemus; tum erit

$$T^2 = \frac{\pi}{2}\,.\,s$$

$$T_1^2 = \frac{2}{\pi}\,s_2$$

Atque primo videamus, quomodo T^1 oriatur e T et T_1^1 e T^1. Erat

$$T = \frac{2^{2n}}{(2n)_n}\;;\qquad T^1 = \frac{2^{2(n+1)}}{\{2\,(n+1)\}_{n+1}}$$

ergo

$$\frac{T^1}{T} = \frac{2^2\,(2n)_n}{\{2\,(n+1)\}_{n+1}}$$

$$= \frac{2^2\,.\,\dfrac{2n\,(2n-1)\,\ldots\,(n+1)}{1.\;2.\;3.\,\ldots\,.\,n}}{\dfrac{(2n+2)\,(2n+1)\,2n\,\ldots\,(n+2)}{1.\,2.\,3.\,\ldots\,.\,(n+1)}}$$

$$= \frac{2^2\,(n+1)}{\dfrac{(2n+2)\,(2n+1)}{n+1}} = \frac{2\,(2n+2)}{(2n+2)}\,\frac{(n+1)}{(2n+1)}$$

$$= \frac{2n+2}{2n+1}$$

$$T^1 = \frac{2n+2}{2n+1}\cdot T$$

qua de re erit

$$T = \frac{2.\,4.\,6.\,8\,\ldots\,2n}{1.\,3.\,5.\,7\,\ldots\,(2n-1)}$$

$$T^2 = \frac{2.\ 2.\ 4.\ 4.\ 6.\ 6.\ 8.\ 8.\ \ldots\ 2n\ .\ 2n}{1.\ 1.\ 3.\ 3.\ 5.\ 5.\ 7.\ 7.\ \ldots\ (2n-1)\ (2n-1)}$$

$$\frac{T^2}{2n} = P_{2n-1}$$

$$\frac{{T^1}^2}{2(n+1)} = P_{2n}$$

T^1 valorem esse reciprocum expressionis
T non erit, qui dubitet; itaque est

$$T_1^1 = \frac{2n+1}{2n+2} \cdot T_1$$

et

$$2n\ T_1^2 = \frac{1}{P_{2n-1}}\ ;\ (2n+1)\ T_1 = \frac{1}{P_{2n}}$$

Jam ad series nostras progrediamur describendas, id quod secundum coëfficientium indefinitorum methodum instituemus. Ponimus primo

$$T_2 = A\ .\ 2n + \frac{B\ .\ 2n}{2n+2} + \frac{C\ .\ 2n}{(2n+2)\ (2n+4)} + \frac{D\ .\ 2n}{(2n+2)\ (2n+4)\ (2n+6)} + \ldots$$

cui seriei justam formam induisse, quandoquidem ejus coëfficientes numeri evadant certi, finiti, non imaginarii, persuasum habemus. Haud difficili transformatione hujus seriei admissa habetur

$$T^2 = A\,2n + B + \frac{C-2B}{2n+2} + \frac{D-4\,C}{(2n+2)\,(2n+4)} + \frac{E-6\,D}{(2n+2)\ (2n+4)\ (2n+6)} + \ldots$$

Qua in serie si $(n+1)$ pro n ponitur, valores quantitati T^1 proprii evadent.

Itaque est

$${T^1}^2 = A\,(2n+2) + B + \frac{C-2B}{2n+4} + \frac{D-4\,C}{(2n+4)\ (2n+6)} + \ldots$$

Jam vero est

$${T^1}^2 = \frac{(2n+2)^2}{(2n+1)^2} \cdot T^2$$

unde

$$(2n+1)\ {T^1}^2 - (2n+2)^2\ T^2 = 0$$

unde consequitur esse

$$2\ {T^1}^2 + (2n+2)\ (T^2 - {T^1}^2) - \frac{{T^1}^2}{2n+2} = 0$$

Ac est

$$T'^2 = 2A(2n+2) + 2B + \frac{2C-4B}{2n+4} + \frac{2D-8C}{(2n+4)(2n+6)} + \dots$$

$$T^2 - T'^2 = -2A + (C-2B)\left\{\frac{1}{2n-2} - \frac{1}{2n-4}\right\}$$

$$+ (D-4C)\left\{\frac{1}{(2n+2)(2n+4)} - \frac{1}{(2n+4)(2n+6)}\right\} + \dots$$

$$= -2A + \frac{2C-4B}{(2n+2)(2n+4)} + \frac{4D-16C}{(2n+2)(2n+4)(2n-6)} + \dots$$

ergo

$$(2n+2)(T^2 - T'^2) = -2A(2n+2) + \frac{2C-4B}{2n+4}$$

$$+ \frac{4D-16C}{(2n+4)(2n+6)} + \frac{6E-36D}{(2n+4)(2n+6)(2n+8)} + \dots$$

et

$$\frac{T'^2}{(2n+2)} = A + \frac{B}{2n+4} + \frac{C}{(2n+4)(2n+6)} + \frac{D}{(2n+4)(2n+6)(2n+8)} + \dots$$

Quas quantitates si conjungamus ratione per aequationem hanc

$$2T'^2 + (2n+2)(T^2 - T'^2) - \frac{T'^2}{2n+2} = 0$$

praescripta, habebimus

$$0 = 2B - A + \frac{4C-9B}{2n+4} + \frac{6D-25C}{(2n+4)(n+6)} + \frac{8E-49D}{(2n+4)(2n+6)(2n+8)} + \dots$$

De hinc

$$2B - A = 0$$
$$4C - 9B = 0$$
$$6D - 25C = 0$$
$$8E - 49D = 0$$

Ergo

$$A = A$$
$$B = \tfrac{1}{2} A$$
$$C = \tfrac{9}{8} A = \frac{3^2}{2.\,4} A$$
$$D = \frac{75}{16} A = \frac{(3.\,5)^2}{2.\,4.\,6} A$$
$$E = \frac{3675}{128} A = \frac{(3.\,5.\,7)^2}{2.\,4.\,6.\,8} A$$

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Unde est

$$T^2 = A \left\{ 2n + \frac{1.\,2n}{2(2n+2)} + \frac{(1.\,3).^2\,2n}{2.4\,(2n-2)\,(2n+4)} + \frac{(1.\,3.\,5)^2\,2n}{2.4.6\,(2n+2)\,(2n+4)\,(2n+6)} + \dots \right\}$$

ubi tantummodo A designandum atque définiendum est.

Jamjam ad tractandam seriem tertiam transeamus, ponamusque

$$T_1^2 = \frac{A}{2n+1} + \frac{B}{(2n+1)\,(2n+3)} + \frac{C}{(2n+1)\,(2n+3)\,(2n+5)} + \dots.$$

$$T_1^{1\,2} = \frac{A}{2n+3} + \frac{B}{(2n+3)\,(2n+5)} + \frac{C}{(2n+3)\,(2n+5)\,(2n+2)} + \dots$$

Ac est

$$T_1^{1\,2} = \frac{(2n+1)^2}{(2n+2)^2} : T_1^2$$

ergo

$$(2n+1)\;(2n+3) \left\{ T_1^2 - T_1^{1\,2} \right\} - 2\,(2n+1)\;T_1 - T_1^{1\,2} = 0$$

Sed est

$$(2n+1)\;(2n+3)\;\left\{ T_1^2 - T_1^{1\,2} \right\}$$

$$= 2A + \frac{4B}{2n+5} + \frac{6C}{(2n+5)\,(2n+7)} + \frac{8D}{(2n+5)\,(2n+7)\,(2n+9)} - \dots$$

$$= 2A + \frac{4B}{2n+3} + \frac{6C - 8B}{(2n+3)\,(2n+5)} + \frac{8D - 24C}{(2n+3)\,(2n+5)\,(2n+7)} + \dots$$

Porro

$$-2\,(2n+1)\;T_1^2$$

$$= -\;2A - \frac{2B}{2n+3} - \frac{2C}{(2n+3)\,(2n+5)} - \frac{2D}{(2n+3)\,(2n+5)\,(2n+7)} - \dots$$

Simili modo erit

$$-T_1^{1\,2} = -\frac{A}{2n+3} - \frac{B}{(2n+3)\,(2n+5)} - \frac{C}{(2n+3)\,(2n+5)\,(2n+7)} - \dots$$

Quas series si secundum rationem per aequationom

$$(2n+1)\;(2n+3)\;\left\{ T_1 - {}^2T_1^{1} \right\} - 2\,(2n+1)\;T_1^2 - T_1^{1\,2} = 0$$

jussam comparemus, habebimus

$$o = \frac{2B - A}{2n+3} + \frac{4C - 9B}{(2n+3)(2n+5)} + \frac{6D - 25C}{(2n+3)(2n+5)(2n+7)} + \cdots$$

Unde consequitur esse

$$\begin{aligned} 2B &- A = o \\ 4C &- 9B = o \\ 6D &- 25C = o \\ 8E &- 49D = o \\ &\ldots \end{aligned}$$

ergo

$$\begin{aligned} A &= A \\ B &= \tfrac{1}{2} A \\ C &= \frac{3^2}{2.\,4} A \\ D &= \frac{(3.\,5)^2}{2.\,4.\,6} A \\ E &= \frac{(3.\,5.\,7)^2}{2.\,4.\,6.\,8} A \\ &\ldots\ldots \end{aligned}$$

Ergo est

$$T_1 \overset{2}{=} A \left\{ \frac{1}{2n+1} + \frac{1}{2(2n+1)(2n+3)} + \frac{(1.\,3)^2}{2.4\,(2n+1)(2n+3)(2n+5)} + \frac{(1.\,3.\,5)^2}{2.\,4.\,6\,(2n+1)\ldots(2n+7)} + \cdots \right\}$$

ubi denuo A accuratius definiendum atque describendum est, id quod in utroque casu facillime procedit. Nam secundum seriem priorem est

$$\frac{T}{2n} \overset{2}{=} A \left\{ 1 + \frac{1}{2(2n+2)} + \cdots \right\} = P_{2n-1}$$

idque pro omni valore literae n tributo. Ponendo $n = \infty$ erit

$$\frac{T}{2n} \overset{2}{=} P_{\infty} = \frac{\pi}{2} = A$$

In altero casu est

$$(2n+1)\, T_1 \overset{2}{=} \frac{1}{P_{2n}} = A \left\{ 1 + \frac{1}{2(2n+3)} + \cdots \right\}$$

ergo pro $n = \infty$ erit

$$(2n+1)\, T_1^2 = \frac{1}{P\infty} = \frac{1}{\frac{\pi}{2}} = A$$

unde sequitur esse

$$A = \frac{2}{\pi}$$

Ergo est

$$T^2 = n\pi \left\{ 1 + \frac{1}{2(2n+2)} + \frac{(1.\ 3)^2}{2.\ 4\ (2n+2)\ (2n+4)} + \frac{(1.\ 3.\ 5)^2}{2.4.6\ (2n+2)\ ..\ (2n+6)} + \ldots \right\}$$

$$T_1^2 = \frac{2}{\pi(2n+1)} \left\{ 1 + \frac{1}{2(2n+3)} + \frac{(1.\ 3)^2}{2.4\ (2n+3)\ (2n+5)} + \frac{(1.\ 3.\ 5)^2}{2.4.6\ (2n+3)\ \ldots\ (2n+7)} + \ldots \right\}$$

Unde series s, s_2 veras et rectas esse videmus.

§. 34.

Si p sit numerus primus, productum $1.2.3.\ldots(p-1)+1$ semper est divisibile sine residuo per p.

Nam secundum praecepta calculi differentialis est:

$$1.2.3.\ldots.\ m = m^m - m_1 (m-1)^m + m_2 (m-2)^m - m_3 (m-3)^m + \ldots + (-1)^{m-1} m_1 \cdot 1^m$$

Denotante p numerum primum $p-1$ par erit, ergo

$$1^0)\quad 1.2.3.\ldots(p-1) = (p-1)^{p-1} - (p-1)_1 (p-2)^{p-1} + (p-1)_2 (p-3)^{p-1} - \ldots + (p-1)_2\, 2^{p-1} - (p-1)\cdot 1^{p-1}$$

Ac jam est

$$2^0)\quad o = (1-1)^{p-1} = 1 - (p-1)_1 + (p-2)_2 - (p-3)_3 + \ldots + (p-1)_2 - (p-1)_1 + 1$$

ergo subtractione facta erit

$$1.2.3..\ (p-1) + 1 = (p-1)^{p-1} - 1 - (p-1)\ \left\{(p-2)^{p-1} - 1\right\} + \ldots + (p-1)_2 \left\{2^{p-1} - 1\right\}$$

Ac vero $p-1$, $p-2$ .... 2 numeri sunt per p non divisibiles, ergo est $(p-1)^{p-1} - 1$; $(p-2)^{p-1} - 1$ .... $2^{p-1} - 1$ secundum theorema Fermatianum per p sine residuo divisibilia, ergo etiam $1.2.3\ldots(p-1)+1$ per p sine residuo divisibile est. Sin vero p sit numerus compositus, tum $1.2.3\ldots(p-1)+1$ per p dividi sine residuo non potest. Quodsi enim est p numerus compositus, factor quidam hujus numeri necessario $\overline{\overline{\angle}}\sqrt{p}$ est; ac jam pro omnibus valoribus numeri $p>3$ est $\sqrt{p}\angle p-1$, factor numeri p ergo necessario inveniri debebit in producto $1.2.3\ldots(p-1)$, ergo et hoc ipsum productum et ipsum p per hunc factorem divisibile est. Jam iu unitatem divisus hic factor communis nihil reliqui non facit, ergo $1.2.3\ldots(p-1)+1$, si respicimus p, est numerus primus ergo non divisibile per p. Huic theorematì nomen est Wilsoniano.

§. 35.

Buzengeigerus in §. 10 dissertationis suae contendit, seriem

$$1.1-\alpha_1\beta_1+\alpha_2\beta_2-\alpha_3\beta_3+\ldots+(-1)^{\alpha}\,1\beta_\alpha$$

simili modo exprimi posse sicuti series

$$1.1+\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3+\ldots+1\beta_\alpha$$

expressionem vero illam pro $1.1.-\alpha_1\beta_1+\ldots+(-1)^{\alpha}\,1.\beta_\alpha=o$ fieri, si α denotet numerum imparem. Quod ut demonstremus, priorem seriem e posteriori derivandam esse judicamus. Alteram seriem dependere ab altera theoremate ab Eulero invento (Calc. Diff. pars II. cap. II. §. 26) nixi demonstrare possumus.

Si

$$S=a+bx+cx^2+dx^3+\ldots$$

sit, semper erit

$$Aa+Bbx+Ccx^2+Ddx^3+Eex^4+\ldots$$

$$=AS+\frac{\triangle AxdS}{1.dx}+\frac{\triangle^2Ax^2d^2S}{1.2.dx^2}+\frac{\triangle^3Sx^3d^3S}{1.2.3.dx^3}+\ldots$$

Si, utentes hoc theoremate, ponamus pro

$$\begin{array}{llllll} a, & b, & c, & d, & e\ \ldots & \text{quantitates} \\ 1, & \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4\ \ldots & \text{et pro} \\ A, & B, & C, & D, & E\ \ldots & \text{quantitates} \\ 1, & -\beta, & \beta_2, & -\beta_3, & \beta_4 & \end{array}$$

erit $S=(1+x)^{\alpha}$

et $-\triangle A=1.1+1_1.\beta_1=(\beta+1)_1$

$$\triangle^2A=1.1+2_1\beta_1+2_2\beta_2=(\beta+2)_2$$

$$-\triangle_3A=1.1+3_1\beta_1+3_2\beta_2+3_3\beta_3=(\beta+3)_3$$

$$- - - - - . - - - - - .$$

$$(-1)^{\alpha}\triangle^{\alpha}A=1.1+\alpha_1\beta_1+\alpha_2\beta_2+\ldots+\alpha_\alpha\beta_\alpha=(\alpha+\beta)_\alpha$$

Ergo est

$$1^0)\quad 1 - \alpha_1\beta_1\, x + \alpha_2\beta_2\, x^2 - \alpha_3\beta_3\, x^3 + \dots + (-1)^\alpha \beta_\alpha\, x^\alpha$$

$$= (1+x)^\alpha - \alpha_1(\beta+1)_1(1+x)^{\alpha-1}x + \alpha_2(\beta+2)_2(1+x)^{\alpha-2}x^2 - \dots$$

$$\dots + (-1)^\alpha \cdot 1.(\alpha+\beta)_\alpha\, x^\alpha.$$

Pro $x = 1$ erit

$$2^0)\quad 1 - \alpha_1\beta_1 + \alpha_2\beta_2 - \alpha_3\beta_3 + \dots + (-1)^\alpha \cdot 1.\beta_\alpha$$

$$= 2^\alpha - 2^{\alpha-1} \cdot \alpha_1(\beta+1)_1 + 2^{\alpha-2} \cdot \alpha_2(\beta+2)_2 - \dots + (-1)^\alpha \cdot (\alpha+\beta)_\alpha$$

Ponendo $\alpha = \beta$ erit

$$1 - \alpha_1^2 + \alpha_2^2 - \alpha_3^2 + \alpha_4^2 - \dots + (-1)^\alpha \cdot 1^2$$

$$= 2^\alpha - 2^{\alpha-1}\alpha_1(\alpha+1)_1 + 2^{\alpha-2}\alpha_2(\alpha+2)_2 - \dots + (-1)^\alpha \cdot (2\alpha)_\alpha$$

Si α denotet numerum imparem, quantitas in sinistra aequationis parte posita $= 0$ erit et habebitur

$$3^0)\quad 2^\alpha - 2^{\alpha-1} \cdot \alpha_1(\alpha+1)_1 - 2^{\alpha-2}\alpha_2(\alpha+1)_2 + \dots - (2\alpha)_\alpha = 0$$

§. 36.

Ut seriem

$$1 - \alpha_1^2 + \alpha_2^2 - \alpha_3^2 + \dots + (-1)^\alpha \cdot 1$$

alio quoque modo exprimere possimus, multiplicemus aequationem hancce

$$(1-x)^\alpha = 1 - \alpha_1 x + \alpha_2 x^2 - \alpha_3 x^3 + \dots + (-1)^\alpha \cdot 1.\, x^\alpha$$

per quantitatem

$$x^{\beta-1}(1-x)^\alpha dx$$

et erit, sumptis integralibus:

$$4^0)\quad S(1-x)^\alpha x^{\beta-1}dx = Sx^{\beta-1}(1+x)^\alpha dx - \alpha_1\, Sx^\beta(1+x)^\alpha dx + \alpha_2\, Sx^{\beta-1}(1+x)^\alpha dx - \dots + (-1)^\alpha Sx^{\beta+\alpha-1}(1+\alpha)^\alpha dx$$

Jam vero in genere, posito $x = 0$ integratione facta, est.

$$5^0)\quad Sx^{\beta+\mu-1}(1+x)^\alpha dx = (-1)^\mu \frac{(\beta+\mu-1)_\mu}{(\beta+\alpha+\mu)_\mu}\, Sx^{\beta-1}(1+x)^\alpha dx$$

Hinc prodit

$$6^0)\quad S(1-x^2)^{\alpha}x^{\beta-1}dx =$$

$$\left\{1+\alpha_1\cdot\frac{\beta_1}{(\beta+\alpha+1)_1}+\alpha_2\cdot\frac{(\beta+1)_2}{(\beta+\alpha+2)_2}+\alpha_3\cdot\frac{(\beta+2)_3}{(\beta+\alpha+3_3}+\ldots\right.$$

$$\left.\ldots+\frac{(\beta+\alpha-1)_\alpha}{(\beta-2\alpha)_\alpha}\right\}\, Sx^{\beta-1}(1+x)^{\alpha}dx$$

Deinde si x = o ponatur est

$$Sx^{\beta-1}(1-x^2)^{\alpha}dx = (-1)^{\alpha}\frac{Sx^{\beta-1}dx}{(\frac{\beta}{2}+\alpha)_\alpha}$$

$$\text{et } Sx^{\beta-1}(1+\alpha)^{\alpha}dx = (-1)^{\alpha}\frac{Sx^{\beta-1}dx}{(\beta+\alpha)_\alpha}$$

Hinc prodit

$$7^0)\quad \frac{(\beta+\alpha)_\alpha}{\left\{\frac{\beta}{2}+\alpha\right\}_\alpha} = 1+\alpha_1\frac{\beta_1}{(\alpha+\beta+1)_1}+\alpha_2\frac{(\beta+2)_2}{(\alpha+\beta+2)_2}+\alpha_3\frac{(\beta+3)_3}{(\alpha+\beta+3)_3}$$

$$+\ldots 1\frac{(\beta+\alpha-1)_\alpha}{(\beta+2\alpha)_\alpha}$$

Si α sit numerus par, erit

$$8^0)\quad \frac{(\alpha+\beta)_\alpha}{(\frac{\beta}{2}+\alpha)_\alpha} = \frac{2^\alpha(\beta+1)(\beta+3)(\beta+5)\ldots(\beta+\alpha-1)}{(\alpha+\beta+2)(\alpha+\beta+4)\ldots(\alpha+\beta+\alpha)}$$

Si α numerum denotet imparem, habebitur

$$9^0)\quad \frac{(\alpha+\beta)_\alpha}{(\frac{\beta}{2}+\alpha)_\alpha} = \frac{2^\alpha(\beta+1)(\beta+3)(\beta+5)\ldots(\beta+\alpha)}{(\alpha+\beta+1)(\alpha+\beta+3)\ldots(\alpha+\beta+\alpha)}$$

Ac jam est in genere

$$(\mu-\alpha)_{\mu+1} = (-1)^{\mu+1}\cdot\alpha_{\mu+1}$$

Qua relatione si utaris, ponendo in 7⁰) pro β quantitatem $-\alpha$, reminiscens semper esse

$$\frac{(\alpha+\beta)_\alpha}{\left\{\frac{\beta}{2}+\alpha\right\}_\alpha} = (-1)^{\frac{\alpha}{2}}\cdot\frac{2^\alpha(\alpha-1)(\alpha-3)\ldots\ldots 3.\ 1}{2.\ 4.\ 6\ldots\ldots\alpha}$$

(in qua relatione α numerus par esse debet), quia quantitas, si α impar esset numerus,

$$\frac{(\alpha+\beta)_\alpha}{(\beta+\alpha)_{\frac{\alpha}{2}}}$$

ut 9°) docet = o esset), habebis

$$10^0)\quad 1 - \alpha_1^2 + \alpha_2^2 - \alpha_3^2 + \ldots + (-1)^\alpha \cdot 1$$

$$= (-1)^{\frac{\alpha}{2}} \cdot \frac{1.\ 3.\ 5.\ 7\ \ldots\ (\alpha-1)}{2.\ 4.\ 6\ \ldots\ \alpha} \cdot 2^\alpha$$

ergo etiam, si scripseris 2α pro α

$$S = 1 - (2\alpha)_1^2 + (2\alpha)_2^2 - \ldots + (-1)^{2\alpha} \cdot 1 = (-1)^\alpha \cdot \frac{1.\ 3.\ 5\ \ldots\ (2\alpha-1)}{2.\ 4.\ 6\ \ldots\ 2\alpha} \cdot 2^{2\alpha}$$

Erat autem

$$S^1 = 1 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \ldots + 1^2 = \frac{1.\ 3.\ 5\ \ldots\ (2\alpha-1)}{2.\ 4.\ 6\ \ldots\ 2\alpha} \cdot 2^{2\alpha}$$

unde perspicuum est, series S et S^1 numericos eosdem habere valores, inter se aequales esse si α in 2α numerum designet parem.

§. 37.

Coroll.

Quoniam

$$1 - (2\alpha)_1^2 + (2\alpha)_2^2 - (2\alpha)_3^2 + \ldots + (-1)^{2\alpha} \cdot 1$$

$$= 2^{2\alpha} - 2^{2\alpha-1}\ (2\alpha)_1\ (2\alpha+1)_1 + 2^{2\alpha-2}\ (2\alpha)_2\ (2\alpha+2)_2 - \ldots + (4\alpha)_{2\alpha}$$

erat, etiam

$$2^{2\alpha} - 2^{2\alpha-1}\ (2\alpha)_1\ (2\alpha+1)_1 + 2^{2\alpha-2}\ (2\alpha)_2\ (2\alpha+2)_2 - 2^{2\alpha-3}\ (2\alpha)_3\ (2\alpha+3)_3$$

$$+ \ldots + (4\alpha)_{2\alpha} = \frac{1.\ 3.\ 5\ \ldots\ (2\alpha-1)}{2.\ 4.\ 6\ \ldots\ 2\alpha} \cdot 2^{2\alpha}$$

erit.