## §. 1.

**Denotante n** litera numerum positivum aut negativum, integrum aut fractum, **m** litera numerum positivum integrum; **uncia $m^{ta}$ seu coëfficiens binomialis $m^{tus}$** quantitas vocatur hacce formula expedita

$$\frac{n\ (n-1)\ \ldots\ (n-m+2)\ (n-m+1)}{1\ .\ 2\ \ldots\ (m-1)\ \ m}$$

Numerum **n exponentem unciae, m** vero numerum **indicem** sive **numerum localem,** sive **signum locale unciae**

$$\frac{n\ (n-1)\ \ldots\ (n-m+1)}{1\ .\ 2\ .\ .\ .\ .\ .\ m}$$

appellare juvabit.

## §. 2.

Jam quantitas, quam §. 1. unciam definivimus, quum in permultis disquisitionibus analyticis saepissime occurrat, viri docti, ut illa peculiari atque proprio charactere exprimeretur, non solum utile sed etiam necessarium esse judicaverunt. Quo factum est, ut complures viri illustrissimi, qui mathesi operam navabant, ejusmodi characteres docuerint, quorum tamen characterum adhuc ab omnibus acceptum videmus nullum.

Hindenburgius a), ut ab hoc exordiar, quem analyseos combinatoriae autorem exstitisse nemo est quem fugiat, ceteri qui ejus disciplinam sequuntur, in quorum numero Eschenbachium b), Burkhardium c), Toepferum d), Rothium e), Kluegelium f), Pfaffium g) laudamus, uncias pro exponente **n** incipientes a prima uncia ita denotabant:

$${}^n\mathfrak{A},\ {}^n\mathfrak{B},\ {}^n\mathfrak{C},\ {}^n\mathfrak{D},\ {}^n\mathfrak{E},\ \ldots$$

ut ex hac uncias significandi ratione sit e. g

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a) Novi systematis Permutationum, Combinationum et Variationum primae lineae Auc. Hindenb. Lips. 1781. p. XL.

b) Eschenbachius: de serierum reversione formulis analytico-combinatoriis exhibita specimen. Lips. 1789.

c) Viro clarissimo H. A. Toepfer summos in Philosophia honores amicorum nomine gratulatur J. C. Burkhard. Inest methodus combinatorio-analytica evolvendis fractionum continuarum valoribus maxime idonea. Lips. 1794. p. XIII.

d) Combinatorische Analysis und Theorie der Dimensionszeichen in Parallele gestellt von H. A. Töpfer. Leipz. 1793.

e) Formulae de serierum reversione demonstratio universalis signis localibus combinatorio-analyticorum vicariis exhibita. Auct. H. A. Rothe. Lipsiae 1793.

f) Sammlung combinatorisch-analytischer Abhandlungen, herausg. v. Hindenburg. Leipz. 1796. Th. I. pag. 48.

g) cf. (f).

$^{n}\mathfrak{K}$

character unciae decimae pro exponente n. Uncia generalis $m^{ta}$ pro exponente n ab Hindenburgio significabatur charactere

$$^{n}\mathfrak{M}$$

Quod tamen signum illud incommodi habet, quod

$$^{n}\mathfrak{M}$$

duodecimam quoque unciam denotat. Fortasse dicet aliquis, incommodum illud, typis si characteres exscribantur, nulla opera posse effugi, adhibitis pro $\mathfrak{M}$ litera signis typographicis diversis, prout

$$^{n}\mathfrak{M}$$

duodecimam aut $m^{tam}$ unciam denotet. Attamen negari nequit, ubi stilo exarandi sint illi characteres, difficilius id esse. Qua de re hanc Hindenburgii uncias indicandi rationem haud aptam et incommodam abjiciendam esse censemus. Sed quia Hindenburgii ceterorumque qui hujus praeclari viri vestigia premebant, praecepta percipere nemo possit, nisi qui hanc uncias denotandi methodum cognoverit; reliquum est, ut dicamus, Hindenburgium, quoties $(m \pm k)^{ta}$ uncia esset exprimenda, toties distantiae signis, quae ipse dicebat, uti, unciamque, quam modo monuimus signo

$$\overset{\pm k}{^{n}\mathfrak{M}}$$

denotare consuevisse.

Alteram uncias notandi rationem Thibaut h) docuit, quae ita comparata est, ut unciae pro exponente n ex ordine a prima uncia efferantur

$$^{n}\overset{1}{\mathfrak{B}},\ ^{n}\overset{2}{\mathfrak{B}},\ ^{n}\overset{3}{\mathfrak{B}},\ ^{n}\overset{4}{\mathfrak{B}},\ \ldots$$

uncia autem $m^{ta}$

$$^{n}\overset{m}{\mathfrak{B}}$$

Cui rationi, quod nos judicamus, illud objiciendum esse videtur, quod litera major $\mathfrak{B}$ saepius adhibita valde incommoda sit, idque maxime si respicis longas expressiones analyticas.

Eulerus i) unciam $m^{tam}$ pro exponente n significat charactere

$$\left[\frac{n}{m}\right]$$

quod tamen symbolum a multis incommodis non abhorrere sole clarius est.

Quartam uncias exprimendi methodum Rothius k) qui primo, ut supra monuimus Hindenburgii signis utebatur, ostendit. Rothio auctore unciae pro exponente n ex ordine a prima uncia denotantur signis:

$$n_1,\quad n_2,\quad n_3,\quad n_4,\quad n_5,\quad \ldots$$

uncia generalis $m^{ta}$ pro exponente n audit

$$n_m.$$

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h) Thibaut Grundriß der allgemeinen Arithmetik oder Analysis. Göttingen 1809.

i) Acta petropolitana 1781. Pars I. pag. 89.

k) Rothe Theorie der combinatorischen Integrale. Nürnberg 1820.

Hanc rationem ceterarum omnium quas tractavimus, simplicissimam esse, non erit qui dubibet; hincque factum est, ut haec ipsa methodus nostris temporibus apud viros doctos maxime invaluerit. Id quod fortassis huic uncias designandi viae verti possit vitio, eo continetur, quod analyticis disquisitionibus amplis et copiosis fieri nequeat, quin symbola ejusdem formae cum modo descripta ad alias quoque quantitates, quae unciae non sint, significandas adhibeantur. Jam vero in explicanda unciarum doctrina, quam hic expositiuri sumus, hoc Rothii signorum vitium, quum facillime effugere possimus, nobiscum constituimus, per totum hoc scriptum nulla alia nisi Rothii simplicissima atque aptissima unciarum significatione uti. Itaque semper unciam mtam pro exponente n significemus charactere

$$n_m$$

i. e.

$$n_m = \frac{n\,(n-1)\ \ldots\ (n-m+1)}{1.\ 2\ \ldots\ m}$$

§. 3.

Denotante n litera quemviscunque positivum aut negativum, integrum aut fractum numerum, k autem litera numerum positivum aut negativum integrum, m litera denique numerum positivum integrum, uncia mta pro exponente n e quantitate

$$\frac{n\,(n+k)\ (n+2k)\ \ldots\ [n+(m-1)k]}{1.\ 2.\ 3.\ \ldots\ m}$$

quam hoc in scripto semper signo

$$\overset{k}{n}_m$$

denotabimus, ita ut sit

$$\text{a)}\quad \overset{k}{n}_m = \frac{n\,(n+k)\ (n+2k)\ \ldots\ [n+(m-1)k]}{1.\ 2.\ 3\ \ldots\ m}$$

procul dubio ita oritur, ut k = — 1 ponatur, quia per hanc substitutionem habebitur

$$\overset{-1}{n}_m = \frac{n\,(n-1)\ (n-2)\ \ldots\ (n-m+1)}{1.\ 2.\ 3.\ \ldots\ m}$$

seu omisso illo — 1, quod in sinistra hujus aequationis supra n litera scriptum est,

$$\text{b)}\quad n_m = \frac{n\,(n-1)\ (n-2)\ \ldots\ (n-m+1)}{1.\ 2.\ 3\ \ldots\ m}$$

Si m literae in aequatione a.) valores 1, 2, 3 ... m, m + 1 successive tribuantur, erit

$$\text{a}_1)\quad \overset{k}{n}_1 = \frac{n}{1}$$

$$\overset{k}{n}_2 = \frac{n\,(n+k)}{1.\ 2.}$$

$$\overset{k}{n}_3 = \frac{n\,(n+k)\ (n+2k)}{1.\ 2.\ 3.}$$

. . . . . .

$$\overset{k}{n}_{m} = \frac{n\,(n+k)\;(n+2k)\;.\;.\;.\;[\,n+(m-1)\,k\,]}{1.\quad 2.\quad 3.\quad .\quad .\quad m}$$

$$\overset{k}{n}_{m+1} = \frac{n\,(n+k)\;(n+2k)\;.\;.\;.\;(n+mk)}{1.\quad 2.\quad 3.\quad .\quad .\quad m+1}$$

E quibus aequationibus, posito $k = -1$ facile consequitur esse

$$\overset{-1}{n}_{1} = \frac{n}{1}$$

$$\overset{-1}{n}_{2} = \frac{n\,(n-1)}{1.\quad 2.}$$

$$\overset{-1}{m}_{3} = \frac{n\,(n-1)\;(n-2)}{1.\quad 2.\quad 3.}$$

— — — — — — — —

$$\overset{-1}{n}_{m} = \frac{n\,(n-1)\;(n-2)\;.\;.\;.\;(n-m+1)}{1.\quad 2.\quad 3.\quad .\quad .\quad m}$$

$$\overset{-1}{n}_{m+1} = \frac{n\,(n-1)\;(n-2)\;.\;.\;.\;(n-m)}{1.\quad 2.\quad 3.\;.\;.\;.\quad m+1}$$

seu omisso charactere $-1$ supra $n$ posito

b₁) $$n_{1} = \frac{n}{1}$$

$$n_{2} = \frac{n\,(n-1)}{1.\quad 2.}$$

$$n_{3} = \frac{n\,(n-1)\;(n-2)}{1.\quad 2.\quad 3.}$$

— — — — — — — —

$$n_{m} = \frac{n\,(n-1)\;(n-2)\;.\;.\;.\;(n-m+1)}{1.\quad 2.\quad 3.\quad .\quad .\quad m}$$

$$n_{m+1} = \frac{n\,(n-1)\;(n-2)\;.\;.\;.\;(n-m)}{1.\quad 2.\quad 3.\quad .\quad .\quad m+1} \quad *)$$

Quae igitur de quantitate $\overset{k}{n}_{m}$ pronunciari poterunt, ea omnia de unciis quoque proferri posse, neminem fugit.

Unde consequitur, omnem unciarum doctrinam nonnisi casum quendam esse specialem doctrinae quae circa quantitatem $\overset{k}{n}_{m}$ exponi possit.

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*) Ubi in hoc scripto brevitatis gratia illud $-1$ omittere placebit, id ita designabimus, ut tali expressioni literam b.) anteponamus.

### §. 4.

Omnem unciam ex antecedente uncia derivari posse demonstratur.

Quoniam secundum §. 3. a.) in genere

$$n_m^k = \frac{n\,(n+k)\,(n+2k)\ldots[n+(m-1)\,k]}{1.\ \ 2.\ \ 3.\ \ .\ \ .\ \ .\ \ m}$$

$$n_{m+1}^k = \frac{n\,(n+k)\,(n+2k)\ \ldots\ (n+mk)}{1.\ \ 2.\ \ 3.\ \ .\ \ .\ \ m+1}$$

est, etiam

a) $$n_{m+1}^k = \frac{n\,(n+k)\,(n+2k)\ldots[n+(m-1)\,k]}{1.\ \ 2.\ \ 3.\ \ .\ \ .\ \ .\ \ m} \cdot \frac{n+mk}{m+1}$$

$$= n_m^k \cdot \frac{n+mk}{m+1}$$

erit. In qua relatione si $k = -1$ ponitur erit

b) $$n_{m+1} = n_m \cdot \frac{n-m}{m+1}$$

Posito pro $m+1$ numero $m$ erit

a₁) $$n_m^k = n_{m-1}^k \cdot \frac{n+(m-1)\,k}{m}$$

b₁) $$n_m = n_{m-1} \cdot \frac{n-m+1}{m}$$

### §. 5.

E §. 4. facile habebitur

a) $$m.\ n_m^k = [n+(m-1)\,k] \cdot n_{m-1}^k$$

b) $$m.\ n_m = (n-m+1) \cdot n_{m-1}$$

### §. 6.

Omnem unciam ex uncia insequente deduci posse demonstratur.

Quia secundum §. 5

$$m \cdot n_m^k = [n+(m-1)\,k] \cdot n_{m-1}^k$$

erat, etiam

a) $$n_{m-1}^k = n_m^k \cdot \frac{m}{n+(m-1)\,k}$$

b) $$n_{m-1} = n_m \cdot \frac{m}{n-m+1}$$

esse debet.

Cui uncias deducendi rationi si in $\overset{k}{n}_1$ quoque tribuerimus locum, erit

a₁) $$\overset{k}{n}_{1-1} = \overset{k}{n}_0 = \overset{k}{n}_1 \cdot \frac{1}{n}$$

b₁) $$n_{n-1} = n_0 = n_1 \cdot \frac{1}{n}$$

Et quia semper secundum §. 3. a₁) et b₁)

$$\overset{k}{n}_1 = \frac{n}{1}\,; \quad n_1 = \frac{n}{1}$$

erat, etiam

a₂) $$\overset{k}{n}_0 = n \cdot \frac{1}{n} = 1$$

b₂) $$n_0 = n \cdot \frac{1}{n} = 1$$

erit.

Ergo pro quovis valore literae n tributo uncia ota aequalis est unitati. Hanc ipsam uncias deducendi viam si unciae otae accomodaverimus, erit

a₃) $$\overset{k}{n}_{0-1} = \overset{k}{n}_{-1} = \overset{k}{n}_0 \cdot \frac{0}{n-1} = 0$$

b₃) $$n_{0-1} = n_{-1} = n_0 \cdot \frac{0}{n} = 0$$

Jam quantitates resp. per $\overset{k}{n}_{-2}$, $\overset{k}{n}_{-3}$ ... et $n_{-2}$, $n_{-3}$ ... signatas $= 0$ esse neminem fugiet, ita ut pro quovis n omnes unciae, quae antecedunt otam, aequales sint 0.

## §. 7.

Si n numerum positivum denotat, est

a) $$\overset{k}{n}_n = \frac{n\,(n+k)\,(n+2k)\dots[n+(n-1)\,k]}{1.\;2.\;3.\;\dots\dots\;n}$$

b) $$n_n = \frac{n\,(n-1)\,(n-2)\dots(n-n+1)}{1.\;2.\;3\;.\;.\;n} = \frac{n\,(n-1)\,(n-2)\dots 2\,.\,1}{1.\;2.\;3\;\dots\;n} = 1$$

et

$$a_1)\quad \overset{k}{n}_{n+1} = \frac{n\,(n+k)\;(n+2k)\ldots(n+nk)}{1.\;\;2.\;\;\;3.\;\;\;.\;\;\;.\;\;\;n+1}$$

$$b_1)\quad n_{n+1} = \frac{n\,(n-1)\;(n-2)\ldots 4.\,3.\,2.\,1.\,0}{1.\;\;2.\;\;\;3\;\ldots\;n\,(n+1)}$$

$$= 0$$

ergo etiam $n_{n+2},\; n_{n+3}\;\ldots,\;\; n_{n+a} = 0$, ubi $a$ numerum positivum integrum, qui 0 excedit, denotat, esse debet. Omnes igitur unciae, quae, si n est numerus positivus integer, unciam $n^{tam}$ insequuntur, aequales sunt 0. Jam si igitur quis omnes uncias inde a $0^{ta}$ in seriem exponere velit, is si n numerus integer positivus sit, seriem habebit finitam, cujus forma erit

$$n_0,\; n_1,\; n_2,\;\ldots\; n_{n-1},\;\; n_n$$

Membrorum hujus seriei numerus adjecta uncia $0^{ta}$ est $= n + 1$.

## §. 8.

Quia $$\overset{k}{n}_m = \frac{n\,(n+k)\;(n+2k)\;\ldots\;[n+(m-1)\,k]}{1.\;\;\;2.\;\;\;3.\;\;\;.\;\;\;.\;\;\;m}$$

$$= \frac{n}{m}\;.\;\frac{(n+k)\;(n+2k)\ldots[n+k+((m-1)-1)\,k]}{1.\;\;\;2.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;m-1}$$

et $$\overset{k}{(n+k)}_{m-1} = \frac{(n+k)\;(n+2k)\ldots[n+k+((m-1)-1)\,k]}{1.\;\;\;2.\;\;.\;\;.\;\;.\;\;.\;\;(m-1)}$$

est, ergo etiam

$$a)\quad \overset{k}{n}_m = \frac{n}{m}\;.\;\overset{k}{(n+k)}_{m-1}$$

$$\text{et}\quad b)\quad n_m = \frac{n}{m}\;.\;(n-1)_{m-1}$$

seu

$$a_1)\quad m.\,\overset{k}{n}_m = n\,\overset{k}{(n+k)}_{m-1}$$

$$b_1)\quad m.\,n_m = n.\;(n-1)_{m-1}$$

esse debet, qua relatione auxiliante uncia $m^{ta}$ pro exponente n semper ex uncia $(m-1)^{ta}$ pro exponente $n-1$ facillime derivari potest.

## §. 9.

Quia

$$\overset{k}{(n+k)}_{m-1} = \frac{(n+k)\;(n+2k)\ldots[n+k+((m-1)-1)\,k]}{1.\;\;\;2.\;\;,\;\;,\;\;,\;\;.\;\;.\;\;(m-1)}$$

$$= \frac{(n+k)\ (n+2k)\ldots[n+(m-1)\,k]}{1.\ \ 2.\ \ .\ \ .\ \ .\ (m-1)}$$

$$\text{et } (\overset{k}{n+k})_m = \frac{(n+k)\ (n+2k)\ \ldots\ (n+mk)}{1.\ \ 2.\ \ .\ \ .\ \ .\ \ m}$$

est, etiam esse

$$(\overset{k}{n+k})_{m-1} + (\overset{k}{n+k})_m = \frac{(n+k)\ (n+2k)\ldots[n+(m-1)\,k]}{1.\ \ 2.\ \ .\ \ .\ \ (m-1)}$$

$$+ \frac{(n+k)\ (n+2k)\ldots(n+(m-1)\,k)}{1.\ \ 2.\ \ .\ \ .\ \ (m-1)}\,\frac{(n+mk)}{m}$$

perspicuum est

a.) $$(\overset{k}{n+k})_{m-1} + (\overset{k}{n+k})_m = \frac{(n+k)\ (n+2k)\ldots[n+(m-1)\,k]}{1.\ \ 2.\ \ .\ \ .\ \ .\ (m-1)} \cdot \frac{m+n+mk}{m}$$

$$= \frac{(n+k)\ (n+2k)\ldots[n+(m-1)\,k]}{1.\ \ 2.\ \ .\ \ .\ \ .\ (m-1)} \cdot \frac{n+(k+1)\,m}{m}$$

$$= \frac{n\,(n+k)\ (n+2k)\ldots[m+(m-1)\,k]}{1.\ \ 2.\ \ 3.\ \ .\ \ .\ \ .\ \ m} \cdot \frac{n+(k+1)\,m}{n}$$

$$= \overset{k}{n}_m \cdot \frac{n+(k+1)\,m}{n}$$

et b.) $$(n-1)_{m-1} + (n-1)_m = n_m \cdot \frac{n}{n}$$

$$= n_m$$

Qua gravissima relatione omnis fere unciarum theoria nititur.

## §. 10.

Jam si relatione §. 9. b.) exhibita ita utaris, ut literae m successive valores 0, 1, 2. 3 . . . m, literae n autem quoslibet valores tribuas, unciarum tabulam facillime condere poteris. Tabulam pro $n = 0{,}01$ usque ad $n = 1{,}00$. Hantschl exhibuit in libro, cui inscriptum est: Logarithmisch-trigonometrisches Handbuch. Wien bei Rohrmann und Schweigert. 1833.

## §. 11.

Si n, m, r numeri sint positivi integri et $m + r = n$ sit, semper

$$n_m = n_r = n_{n-m}$$

$$\text{Nam } n_m = \frac{n\,(n-1)\ldots(n-m+1)}{1.\ \ 2.\ \ .\ \ .\ \ .\ \ m}$$

$$n_r = \frac{n\,(n-1)\ldots(n-r+1)}{1.\ \ 2.\ \ .\ \ .\ \ .\ \ r}$$

est. Quum vero ex hypothesi $m + r = n$ esset, etiam $n - m = r$, $n - r = m$ est.

Unde perspicuum est consequi

$$n_m = \frac{n(n-1)\ldots(r+2)(r+1)}{1.\ 2.\ .\ .\ .\ .\ .\ m}$$

$$n_r = \frac{n(n-1)\ldots(m+2)(m+1)}{1.\ 2.\ .\ .\ .\ .\ .\ r}$$

Jam vero per se patet esse

$$1.\ 2.\ 3\ldots r \times (r+1)(r+2)\ldots(n-1)\,n$$
$$= 1.\ 2.\ 3\ldots m \times (m+1)(m+2)\ldots(n-1)\,n$$

seu

$$1.\ 2.\ 3\ldots r \times n(n-1)\ldots(r+2)(r+1)$$
$$= 1.\ 2.\ 3\ldots m \times n(n-1)\ldots(m+2)(m+1)$$

ergo

$$\frac{n(n-1)\ldots(r+1)}{1.\ 2.\ .\ .\ .\ m} = \frac{n(n-1)\ldots(m+1)}{1.\ 2.\ .\ .\ r}$$

id est,

$$n_m = n_r = n_{n-m}$$

## §. 12.

Sit primo $n = 2\,m$ numerus positivus integer par, unciae pro hoc exponente erunt

$$n_0,\ n_1 \times n_2,\ n_3,\ \ldots\ n_{m-1},\ n_m,\ n_{m+1},\ \ldots\ n_{n-1},\ n_n$$

Atque jam, quum

$$0 + n = n,$$
$$1 + (n-1) = n,$$
$$2 + (n-2) = n$$
$$- \quad - \quad - \quad -$$
$$(m-1) + (m+1) = 2\,m = n$$

sit, etiam est

$$n_{m+1} = n_{m-1}$$
$$- \quad - \quad - \quad - \quad -$$
$$n_{n-2} = n_2$$
$$n_{n-1} = n_1$$
$$n_n = n_0$$

seriesque unciarum modo exposita hac quoque ratione depingi potest:

$$^{n}0,\ ^{n}1,\ ^{n}2,\ \ldots\ ^{n}m-1,\ ^{n}m,\ ^{n}m+1,\ \ldots\ ^{n}2,\ ^{n}1,\ ^{n}0,$$

seu etiam sequenti modo

$$^{n}0,\ ^{n}1,\ ^{n}2,\ ^{n}{}_{\frac{1}{2}n-1},\ ^{n}{}_{\frac{1}{2}n},\ ^{n}{}_{\frac{1}{2}n-1},\ \ldots\ ^{n}2,\ ^{n}1,\ ^{n}0,$$

Sit porro $n = 2m - 1$ numerus positivus integer impar, series unciarum pro hoc exponente erit

$$^{n}0,\ ^{n}1,\ ^{n}2,\ \ldots\ ^{n}m-2,\ ^{n}m-1,\ ^{n}m,\ ^{n}m+1,\ \ldots\ ^{n}n-1,\ ^{n}n$$

Ac jam quum sit

$$\begin{array}{ccccccc} 0 & + & n & = & n & & \\ 1 & + & (n-1) & = & n & & \\ 2 & + & (n-2) & = & n & & \\ - & - & - & - & - & - & - \\ (m-2) & + & (m+1) & = & 2m-1 & = & n \\ (m-1) & + & m & = & 2m-1 & = & n \end{array}$$

etiam

$$\begin{array}{ccc} ^{n}m & = & ^{n}m-1 \\ ^{n}m+1 & = & ^{n}m-2 \\ - & - & - \\ ^{n}n-2 & = & ^{n}2 \\ ^{n}n-1 & = & ^{n}1 \\ ^{n}n & = & ^{n}0 \end{array}$$

esse debet, unde consequitur seriem nostram unciarum hac quoque exprimi posse ratione

$$^{n}0,\ ^{n}1,\ ^{n}2,\ \ldots\ ^{n}m-2,\ ^{n}m-1,\ ^{n}m-1,\ ^{n}m-2,\ \ldots\ ^{n}2,\ ^{n}1,\ ^{n}0$$

seu etiam sequenti modo

$$^{n}0,\ ^{n}1,\ ^{n}2,\ \ldots\ ^{n}{}_{\frac{1}{2}(n-1)},\ ^{n}{}_{\frac{1}{2}(n-1)}\ \ldots\ ^{n}2,\ ^{n}1,\ ^{n}0$$

## §. 13.

E §. 9. concluditur esse

$$1_0)\quad (n-1)_{m-1}, = {}^{n}m - (n-1)_m$$

$$2_0)\quad (n-1)_m = {}^{n}m - (n-1)_{m-1}$$

## §. 14.

Qui membra dissolvendi methodum §. 9. descriptam, si n numerum positivum integrum designet, longius extenderit, habebit

$$(n+1)_{m+1} = n_m + n_{m+1}$$
$$= n_m + (n-1)_m + (n-1)_{m+1}$$
$$= n_m + (n-1)_m + (n-2)_m + (n-2)_{m+1}$$

— — — — — — — — —

$$= n_m + (n-1)_m + (n-2)_m + \cdot\cdot + 2_m + 1_m + 0_m + 0_{m+1}$$

seu, quoniam $0_{m+1}$ pro omnibus valoribus literae m subjunctis

$= 0$ sit

$$(n+1)_{m+1} = n_m + (n-1)_m + (n-2)_m + \cdot\cdot + 2_m + 1_m + 0_m$$

Quae series audiet pro

$$m = 0$$
$$(n+1)_1 = n_0 + (n-1)_0 + (n-2)_0 + \cdot\cdot + 2_0 + 1_0 + 0_0$$
$$m = 1$$
$$(n+1)_2 = n_1 + (n-1)_1 + (n-2)_1 + \cdot\cdot + 2_1 + 1_1$$
$$m = 2$$
$$(n+1)_3 = n_2 + (n-1)_2 + (n-2)_2 + \cdot\cdot + 3_2 + 2_2$$

— — — — — — —

$$m = m$$
$$(n+1)_{m+1} = n_m + (n-1)_m + (n-2)_m + \cdot\cdot + (m+1)_m + m_m$$

Hanc supremo loco exaratam seriem pro $m = m$, ubi ordine inverso scripseris habebis

$$(n+1)_{m+1} = m_m + (m+1)_m + (m+2)_m + \cdot\cdot\cdot + (n-1)_m + n_m$$

Ac vero quum cf. §. 11.

$$m_m = m_0$$
$$(m+1)_m = (m+1)_1$$

$$(m+2)_m = (m+2)_2$$

$$(m+3)_m = (m+3)_3$$

— — — — —

$$(n-1)_m = (n-1)_{n-m-1}$$

$$n_m = n_{n-m}$$

sit, non erit, qui neget esse

$$(n+1)_{m+1} = m_0 + (m+1)_1 + (m+2)_2 + (m+3)_3 + \cdot\cdot + (n-1)_{n\text{-}m\text{-}1} + n_{n-m}$$

i. e.

$$\begin{aligned}(n+1)_{m+1} = {} & 1 \\ & + \frac{m+1}{1} \\ & + \frac{(m+1)\ (m+2)}{1.\ \ 2.} \\ & + \frac{(m+1)\ (m+2)\ (m+3)}{1.\ \ 2.\ \ 3.} \\ & - \ - \ - \ - \ - \\ & + \frac{(m+1)\ (m+2)\ \cdots\ (n-1)}{1.\ \ 2.\ \ .\ \ .\ \ (n\text{-}m\text{-}1)} \\ & + \frac{(m+1)\ (m+2)\ \ldots\ (n-1)\ n}{1.\ \ 2.\ \ .\ \ .\ \ (n-m)}\end{aligned}$$

## §. 15.

Aequatio §. 14 explicata

$$(m+1)_{m+1} = m_m + (m+1)_m + (m+2)_m + \ldots + (n-1)_m + n_m$$

ansam praebet ea, quae sequuntur observandi. Scribendo sc. successive pro m valores 1, 2, 3, 4 . . . conformabitur haec series ita, ut sit pro

1°) $m = 1$

$$1_1 + 2_1 + 3_1 + \cdots + (n-1)_1 + n_1 = (n+1)_2$$

i. e.

$$\frac{1}{1} + \frac{2}{1} + \frac{3}{1} + \frac{4}{1} + \cdot\cdot + \frac{n-1}{1} + \frac{n}{1} = \frac{n\,(n+1)}{1.\ \ 2.}$$

2°) $m = 2$

$$2_2 + 3_2 + 4_2 + \cdots + n_2 + (n+1)_2 = (n+2)_3$$

i. e.

$$\frac{1.\,2}{1.\,2} + \frac{2.\,3}{1.\,2} + \frac{3.\,4}{1.\,2} + \frac{4.\,5}{1.\,2} + \cdot\cdot + \frac{n\,(n+1)}{1.\;\;2.} = \frac{n\,(n+1)\;(n+2)}{1.\;\;2.\;\;\;3.}$$

3°) $m = 3$

$$3_3 + 4_3 + 5_3 + 6_3 + \cdot\cdot + (n+1)_3 + (n+2)_3 = (n+3)_4$$

i. e.

$$\frac{1.\,2.\,3}{1.\,2.\,3} + \frac{2.\,3.\,4}{1.\,2.\,3} + \frac{3.\,4.\,5}{1.\,2.\,3} + \cdot\cdot + \frac{n\,(n+1)\,(n+2)}{1.\;\;2.\;\;3.} = \frac{n\,(n+1)\;(n+2)\;(n+3)}{1.\;\;2.\;\;\;3.\;\;\;4.}$$

4°) $m = 4$

$$4_4 + 5_4 + 6_4 + 7_4 + \cdot\cdot + (n+2)_4 + (n+3)_4 = (n+4)_5$$

i. e.

$$\frac{1.\,2.\,3.\,4}{1.\,2.\,3.\,4} + \frac{2.\,3.\,4.\,5}{1.\,2.\,3.\,4} + \frac{3.\,4.\,5.\,6}{1.\,2.\,3.\,4} + \cdot\cdot + \frac{n\,(n+1)\;(n+2)\;(n+3)}{1.\;\;\;2.\;\;\;3.}$$

$$= \frac{n\,(n+1)\;(n+2)\;(n+3)\;(n+4)}{1.\;\;2.\;\;\;3.\;\;\;4.\;\;\;5}$$

Hac eadem ratione quomodo progrediendum sit, dijudicare non haerebis.

Singula harum serierum membra **numeri figurati** vocantur. Membra seriei 1°, nominantur numeri figurati **primi ordinis**; membra seriei 2°, dicuntur numeri figurati **secundi ordinis**, seu numeri **trigonales**; membra seriei 3°) appellantur numeri figurati **tertii ordinis** seu numeri **pyramydales**.

**Nota.** Alia methodus ad eosdem numeros figuratos perveniendi aliis unciarum proprietatibus nixa in §. 30. adumbrabitur.

## §. 16.

Secundum §. 9. erat

$$(n+1)_{m+1} = n_m + n_{m+1}$$

Dissolvendo utroque membro dextera aequationis parte posito, habebitur

$$(n+1)_{m+1} = n_m + n_{m+1}$$

$$= (n-1)_{m-1} + (n-1)_m + (n-1)_m + (n-1)_{m+1}$$

$$= (n-1)_{m-1} + 2\,(n-1)_m + (n-1)_{m+1}$$

$$= (n-2)_{m-2} + (n-2)_{m-1} + 2\,(n-2)_{m-1} + 2\,(n-2)_m + (n-2)_m + (-2)_{m+1}$$

$$= (n-2)_{m-2} + 3\,(n-2)_{m-1} + 3\,(n-2)_{m} + (n-2)_{m+1}$$

$$= (n-3)_{m-3} + (n-3)_{m-2} + 3\,(n-3)_{m-2} + 3\,(n-3)_{m-1} + 3\,(n-3)_{m-1} + 3\,(n-3)_{m} + (n-3)_{m} + (n-3)_{m+1}$$

$$= (n-3)_{m-3} + 4\,(n-3)_{m-2} + 6\,(n-3)_{m-1} + 4\,(n-3)_{m} + (n-3)_{m+1}$$

Quam singula hujus seriei membra legem sequantur, nemo erit quin videat.

Eulerus has ipsas relationes insequenti ratiocinatione explicat. Postquam aequationem

$$(p + 1)_q = p_q + p_{q-1}$$

veram esse docuit, ita pergit:

Si loco q scribamus q + 1, formulis permutatis erit

$$p_q + p_{q+1} = (p+1)_{q+1}$$

similique modo numerum q continuo unitate augendo, erit etiam ut sequitur

$$p_{q+1} + p_{q+2} = (p+1)_{q+2}$$

$$p_{q+2} + p_{q+3} = (p+1)_{q+3}$$

$$p_{q+3} + p_{q+4} = (p+1)_{q+4}$$

$$p_{q+4} + p_{q+5} = (p+1)_{q+5}$$

— — — — — —

Quodsi harum aequationum binas se insequentes addamus, prodibunt istae novae aequationes

$$p_q + 2p_{q+1} + p_{q+2} = (p+1)_{q+1} + (p+1)_{q+2} = (p+2)_{q+2}$$

$$p_{q+1} + 2p_{q+2} + p_{q+3} = (p+1)_{q+2} + (p+1)_{q+3} = (p+2)_{q+3}$$

$$p_{q+2} + 2p_{q+3} + p_{q+4} = (p+1)_{q+3} + (p+1)_{q+4} = (p+2)_{q+4}$$

$$p_{q+3} + 2p_{q+4} + p_{q+5} = (p+1)_{q+4} + (p+1)_{q+5} = (p+2)_{q+5}$$

— — — — — — — —

Quodsi denuo binas harum aequationum se insequentes addamus, reperiemus primo

$$p_q + 3p_{q+1} + 3p_{q+2} + p_{q+3} = (p+2)_{q+2} + (p+2)_{q+3} = (p+3)_{q+3}$$

$$p_{q+1} + 3p_{q+2} + 3p_{q+3} + p_{q+4} = (p+2)_{q+3} + (p+2)_{q+4} = (p+3)_{q+4}$$

$$p_{q+2} + 3p_{q+3} + 3p_{q+4} + p_{q+5} = (p+2)_{q+4} + (p+2)_{q+5} = (p+3)_{q+5}$$

— — — — — —

parique modo progredi licebit, quousque libuerit.

Quodsi denuo harum aequationum binas se insequentes addamus, habebimus aequationem hanc

$$p_q + 4p_{q+1} + 6p_{q+2} + 4p_{q+3} + p_{q+4} = (p+4)_{q+5}$$

— — — — —

similique modo erit

$$p_q + 5p_{q+1} + 10p_{q+2} + 10p_{q+3} + 5p_{q+4} + p_{q+5} = (p+5)_{q+6}$$

— — — — —

Hujus seriei per inductionem inventae demonstrationem universalem ejusque usum in demonstrando theoremate quodam gravissimo §. 30. explicabimus.

## §. 17.

Series

$$(n+1)_{m+1} = n_m + (n-1)_m + (n-2)_m + (n-3)_m + \dots$$

definita est, quando n numerum integrum positivum designat et $m = n$ fit. Terminum hujus seriei supremum $(n-m+1)^{tum}$ esse quum manifestem sit, habebit illa series a fine usque ad $n_m$ computando terminos $n-m+1$.

Sin designet n numerum positivum aut negativum fractum aut numerum integrum negativum, haec series infinita erit, immo quantitatas $(n-p)_{m+1}$ seriei addenda est, ubi $p + 1$ indicem ejus termini denotat, quocum series finitur.

Est enim

$$\begin{aligned}
(n+1)_{m+1} &= n_m + n_{m+1} \\
&= n_m + (n-1)_m + (n-1)_{m+1} \\
&= n_m + (n-1)_m + (n-2)_m + (n-2)_{m+1}
\end{aligned}$$

— — — —

$$= n_m + (n-1)_m + (n-2)_m + \dots + (n-k)_m + (n-k)_{m+1}$$

Quodsi igitur seriem istam termino $(n+k)_m$ finire placuerit, complementum $(n-k)_{m+1}$ ei addendum esse per se patet

Ut quae modo exposuimus clariora evadant, addamus exempla nonnulla. Ponamus igitur $n = \frac{1}{2}$, $m = 3$, ita ut $m + 1 = 4$ sit, tum erit

$$\begin{aligned}
(n+1)_{m+1} &= (\tfrac{1}{2}+1)_4 = (\tfrac{3}{2})_4 \\
&= (\tfrac{1}{2})_3 + (-\tfrac{1}{2})_3 + (-\tfrac{3}{2})_3 + (-\tfrac{5}{2})_3 + (-\tfrac{5}{2})_4
\end{aligned}$$

$$= \frac{1.\,1.\,3}{2.\,4.\,6} - \frac{1.\,3.\,5}{2.\,4.\,6} - \frac{3.\,5.\,7}{2.\,4.\,6} - \frac{5.\,7.\,9}{2.\,4.\,6} - \frac{3.\,5.\,7.\,9.\,11}{2.\,4.\,6.\,8}$$

sive

$$= (\tfrac{1}{2})_3 + (-\tfrac{1}{2})_3 + (-\tfrac{3}{2})_3 + (-\tfrac{5}{2})_3 + (-\tfrac{7}{2})_3 + (-\tfrac{7}{2})_4$$

$$= \frac{1.\,1.\,3}{2.\,4.\,6} - \frac{1.\,3.\,5}{2.\,4.\,6} - \frac{3.\,5.\,7}{2.\,4.\,6} - \frac{5.\,7.\,9}{2.\,4.\,6} - \frac{7.\,9.\,11}{2.\,4.\,6} + \frac{7.\,9.\,11.\,13}{2.\,4.\,6.\,8}$$

seu

$$= (\tfrac{1}{2})_3 + (-\tfrac{1}{2})_3 + (-\tfrac{3}{2})_3 + (-\tfrac{5}{2})_3 + (-\tfrac{7}{2})_3 + (-\tfrac{9}{2})_3 + (-\tfrac{9}{2})_4$$

$$= \frac{1.\,1.\,3}{2.\,4.\,6} - \frac{1.\,3.\,5}{2.\,4.\,6} - \frac{3.\,5.\,7}{2.\,4.\,6} - \frac{5.\,7.\,9}{2.\,4.\,6} - \frac{7.\,9.\,11}{2.\,4.\,6} - \frac{9.\,11.\,13}{2.\,4.\,6} + \frac{9.\,11.\,13.\,15}{2.\,4.\,6.\,8}$$

Hac ratione usi seriei quotcunque placuerit terminos tribuere nulla opera possimus.

Sit porro $n = \frac{1}{3}$, $m = 3$, ergo $m + 1 = 4$ et $n + 1 = \frac{4}{3}$

erit

$$(n+1)_{m+1} = (\tfrac{4}{3})_4$$

$$= (\tfrac{1}{3})_3 + (-\tfrac{2}{3})_3 + (-\tfrac{5}{3})_3 + (-\tfrac{8}{3})_3 + (-\tfrac{8}{3})_4$$

$$= \frac{1.\,2.\,5}{3.\,6.\,9} - \frac{2.\,5.\,8}{3.\,6.\,9} - \frac{5.\,8.\,11}{3.\,6.\,9} - \frac{8.\,11.\,14}{3.\,6.\,9} - \frac{8.\,11.\,14.\,17}{3.\,6.\,9.\,12}$$

seu

$$= (\tfrac{1}{3})_3 + (-\tfrac{2}{3})_3 + (-\tfrac{5}{3})_3 + (-\tfrac{8}{3})_3 + (-\tfrac{11}{3})_3 + (-\tfrac{11}{3})_4$$

$$= \frac{1.\,2.\,5}{3.\,6.\,9} - \frac{2.\,5.\,8}{3.\,6.\,9} - \frac{5.\,8.\,11}{3.\,6.\,9} - \frac{8.\,11.\,14}{3.\,6.\,9} - \frac{11.\,14.\,17}{3.\,6.\,9} + \frac{11.\,14.\,17.\,20.}{3.\,6.\,9.\,12}$$

Hunc ipsam ad modum longius quomodo progrediendum sit neminem fugiet.

## §. 18.

Quia

$$(n+1)_{m+1} = n_m + n_{m+1}$$

est, etiam, pro m successive valores 0, 1, 2, 3 . . . m − 1, m, m + 1 ponendo, erit

$$(n+1)_1 = 1 + n_1 = (n+1)\,.\,1$$

$$(n+1)_2 = n_1 + n_2 = \frac{n}{1} + \frac{n\,(n-1)}{1.\;2.} = \frac{n\,(2+n-1)}{1.\;2.} = \frac{n+1}{2}\,.\;n_1$$

$$(n+1)_3 = n_2 + n_3 = \frac{n(n-1)}{1.\ 2.} + \frac{n(n-1)(n-2)}{1.\ 2.\ 3} = \frac{n(n-.)}{1.\ 2.}\left[\frac{3+n-2}{3}\right] = \frac{n+1}{3} \cdot n_2$$

$$(n+1)_4 = n_3 + n_4 = \frac{n(n-1)(n-2)}{1.\ 2.\ 2} + \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4.} = \frac{n(n-1)(n-2)}{1.\ 2.\ 3.}\left[\frac{4+n-3}{4}\right] = \frac{n+1}{4} \cdot n_3$$

— — — — — — —

$$(n+1)_m = n_{m-1} + n_m = \frac{n(n-1)..(n-m+2)}{1.\ 2.\ \dots(m-1)} + \frac{n(n-1)..(n-m+1)}{1.\ 2\ ..\ m} = \frac{n(n-1)..(n-m+2)}{1.\ 2\ ..\ (m-1)}\left[\frac{m+n-m+1}{m}\right]$$

$$= \frac{n+1}{m} \cdot n_{m-1}$$

$$(n+1)_{m+1} = n_m + n_{m+1} = \frac{n(n-1)..(n-m+1)}{1.\ 2.\ ..\ m} + \frac{n(n-1)..(n-m)}{1.2\ ..\ (m+1)} = \frac{n(n-1)..(n-m+1)}{1.\ 2.\ \dots\ m}\left[\frac{m+1+n-m}{m+1}\right]$$

$$= \frac{n+1}{m+1} \cdot n_m$$

Hinc pro quovis literae n tributo valore prodibunt aequationes hae:

$$(n+1)1 = 1\,(n+1)$$

$$(n+1)n_1 = 2\,(n+1)_2$$

$$(n+1)n_2 = 3\,(n+1)_3$$

$$(n+1)n_3 = 4\,(n+1)_4$$

— — — — —

$$(n+1)n_{m-1} = m\,(n+1)_m$$

$$(n+1)n_m = (m+1)\,(n+1)_{m+1}$$

Quum vero sit

$$1 + n_1 + n_2 + n_3 + n_4 + \dots$$

$$= (1+1)^n = 2^n$$

etiam

$$(n+1)2^n = 1.(n+1)_1 + 2(n+1)_2 + 3(n+1)_3 + 4(n+1)_4 + \dots$$

esse debet.

Si igitur pro n numerum positivum integrum ponamus, erit:

$$(n+1)2^n = 1.(n+1)_1 + 2(n+1)_2 + 3(n+1)_3 + \dots + n(n+1)_n + (n+1)(n+1)_{n+1}$$

$$= (n+1)\,[1 + n_1 + n_2 + n_3 + \dots + n_1 + 1]$$

i. e.

$$2^n = 1 + n_1 + n_2 + n_3 + \ldots + n_1 + 1$$

## §. 19.

Uncias exponentium integrorum positivorum et negativorum semper esse numeros **integros** demonstratur.

1°) In genere erat cf. §. 16.

$$\begin{aligned} n_m &= (n-1)_{m-1} + (n-1)_m \\ &= (n-2)_{m-2} + 2(n-2)_{m-1} + (n-2)_m \\ &= (n-3)_{m-3} + 3(n-3)_{m-2} + 3(n-3)_{m-1} + (n-3)_m \end{aligned}$$

— — — —

Quam membrorum solvendorum rationem continuando quantitates habebuntur hac formula expeditae

$$g\,(n-a)_b$$

ubi per a, b, g numeri **positivi integri** denotentur, quorum a et b omnes valores inde a o usque ad m habere possunt, et a semper tanto augeri potest, ut $n-a=b+1$ oriatur. Jam terminus generalis formula $(b+1)_b$ exarari poterit, quae quantitas aequalis est $b+1$, i. e. numero integro aequalis. Hinc $n_m$ esse **summam numerorum** positivorum integrorum intelligitur, ergo ipsum $n_m$ numerum positivum integrum esse perspicitur. Productum, unde numerator fractionis $n_m$ compositus est, constat ex m numeris se insequentibus integris, quorum ordo pro libitu scribi potest; ergo numerator per productum e numeris 1, 2, 3, .. m se insequentibus sine residuo divisibilis est, unde jam nulla opera obtinetur, uncias **etiam pro exponentibus negativis integris**, quum formula unciarum pro exponentibus negativis sit

$$\frac{(-a)\,(-a-1)\,(-a-2)\,..\,(-a-n+1)}{1.\;2.\;3.\;\ldots\;n}$$

sive

$$\frac{a\,(a+1)\,(a+2)\,..\,(a+n-1)}{1.\;2.\;3\;\ldots\;n}\cdot(-1)^n,$$

ubi a numerum integrum designet, semper esse numeros **integros**.

Quod supremo loco posuimus exemplo illustrare juvat. Sit $(1+1)^{-5}$ datum, ubi n numerus negativus integer est. Est

$$(1+1)^{-5} = 1 - \frac{5}{1} + \frac{-5.-6}{1.\;2} + \frac{-5.-6.-7}{1.\;2.\;3.} + \frac{-5.-6.-7.-8}{1.\;2.\;3.\;4}$$

$$+ \frac{-5. -6. -7. -8. -9}{1. \; 2. \; 3. \; 4. \; 5} + \; . \; . \; . \; .$$

$$= 1 - 5 + 15 - 35 + 70 - 126 + 210 - 330 + \, . \, .$$

Quorum numerorum summa est, quoniam

$$(1+1)^{-5} = \frac{1}{(1+1)^5} = \frac{1}{1+5+10+10+5+1}$$

esse debet,

$$= \frac{1}{2^{\,5}} = \frac{1}{32.}$$

2°) Idem theorema etiam hoc modo demonstrari potest.

Quia, si n et m numeros positivos integros denotent, quantitas

$$\frac{n\,(n-1)\;\ldots\;(n-m+1)}{1.\;2.\;.\;.\;.\;.\;.\;m}$$

nihil aliud sit, nisi summa combinationum $m^{tae}$ classis pro n elementis, repetitionibus omissis, haec autem summa e combinationum natura nonnisi numerus positivus integer esse possit; inde etiam quantitatem nostram

$$\frac{n\,(n-1)\;\ldots\;(n-m+1)}{1.\;2.\;.\;.\;.\;.\;.\;m}$$

numerum positivum integrum esse oportere facile colligitur.

Quum porro

$$(-n)_m = \frac{(-n)\,(-n-1)\,(-n-2)\,.\,.\,(-n-m+1)}{1.\;\;2.\;\;3.\;\;.\;\;.\;\;m}$$

$$= \frac{n\,(n+1)\,(n+2)\;\ldots\;(n+m-1)}{1.\;\;2.\;\;.\;\;.\;\;.\;\;m}\,.\,(-1)^m$$

sit et

$$\frac{n\,(n+1)\,(n+2)\,.\,.\,(n+m-1)}{1.\;\;2.\;\;3.\;.\;.\;.\;m}$$

summam combinationum $m^{tae}$ classis pro n elementis repetitionibus admissis designet, ergo

$$\frac{n\,(n+1)\;\ldots\;(n+m-1)}{1.\;2.\;.\;.\;.\;m}$$

$$\text{et } (-n)_m = \frac{n\,(n+1)\ldots(n+m-1)}{1.\;2.\;\;\;m}\,.\,(-1)^m$$

numerus integer esse debet.

## §. 20.

Sit $n = 2p$ numerus positivus integer par, tum numerantur a $0^{ta}$ usque ad $(2p)^{tam}$ unciae $2p+1$; hoc in casu una tantum media uncia apparet, quae $p^{ta}$ est. Sin vero $n = 2p-1$ numerum positivum integrum imparem denotet, tum numerantur $2p$ unciae; hoc igitur in casu aut nulla uncia media aut sunt unciae mediae duae, quae scribuntur

$$(2p-1)_{p-1} \quad \text{et} \quad (2p-1)_p$$

Jam secundum §. 12

$$(2p-1)_{p-1} = (2p-1)_p$$

est. Hanc ipsam quantitatem pro exponentibus imparibus unciam mediam appellabimus.

## §. 21.

Unciam mediam omnium unciarum maximam esse contendimus. Quod theorema ut dilucide demonstremus duos nobis ob oculos ponamus casus, prout $n$ numerus sit par aut impar.

1°) $n$ sit numerus par. Demonstrandum est

$$(2p)_p > (2p)_{p \pm a}$$

seu

$$\frac{(2p)_p}{(2p)_{p \pm a}} > 1$$

esse, ubi $a$ semper numerum positivum integrum, qui 0 excedat significet. Jam est:

$$\frac{(2p)_p}{(2p)_{p\pm a}} = \frac{\dfrac{2p\,(2p-1)\ldots(p+2)\,(p+1)}{1.\ 2.\ 3\ .\ .\ .\ (p-1)p}}{\dfrac{2p\,(2p-1)\ldots(p\mp a+2)\,(p\mp a+1)}{1.\ 2.\ 3\ .\ .\ (p\pm a-1)\,(p\pm a)}}$$

$$= \frac{(p+1)\,(p+2)\ldots(p+a)}{(p-a+1)\,(p-a+2)\ldots p}$$

Deinde est

$$p+1 > p-a+1;$$

$$p+2 > p-a+2$$

$$- \quad - \quad - \quad -$$

$$p+a > p,$$

singuli igitur numeratoris factores majores sunt singulis denumeratoris factoribus, numerator ergo major est denominatore i. e. fractio major est unitate. Quibus dictis, quae enuntiavimus, demonstrata sunt, seu

$$\frac{(2p)_p}{(2p)_{p\pm a}} > 1$$

i. e. $(2p)_p > (2p)_{p\pm a}$

est.

2°) Si n numerus impar sit e. g. $n = 2p-1$, secundum eandem concludendi rationem erit

$$\frac{(2p-1)_{p-1}}{(2p-1)_{p-1-a}} = \frac{(2p-1)_p}{(2p-1)_{p+a}} = \frac{(p+1)\,(p+2)\ldots(p+a)}{(p-a)\,(p-a+1)\ldots p-1)} > 1.$$

## §. 22.

Ad computandum unciae mediae pro exponentibus positivis integris valorem nimis lata multiplicatione, si exponentes numeri essent magni, utendum esset. Sin vero in computanda uncia media nihil aliud spectatur, nisi ut ejus valor evadat approximativus, tum expressiones approximativae pro hac uncia sufficiunt, cujusmodi expressiones in sequentibus tribus §§. docebuntur. Ut quantum expressio approximativa aberret a veritate, perspici possit, de expressione Wallisiana

$$\frac{\pi}{2} = \frac{2.\ 2.\ 4.\ 4.\ 6.\ 6.\ 8.\ 8.\ 10.\ 10}{1.\ 3.\ 3.\ 5.\ 7\ \ 7.\ 9.\ 9.\ 11.\ 11}\ \ldots \text{ in infin.}$$

nonnulla praemittenda sunt.

## §. 23.

Lemma. Jam primo quaeritur, quomodo se habeat productum e numero factorum seriei pro $\frac{\pi}{2}$ erutae finito compositum ad justum $\frac{\pi}{2}$ valorem. Productum igitur e $2n-1$ et $2n$ primis terminis seriei infinitae, de qua sermo est, constans per $P_{2n-1}$ et $P_{2n}$ designare et valorem partialem resp. $(2n-1)^{\text{tum}}$ et $2n^{\text{tum}}$ seriei nostrae dicere liceat, ita ut sit

$$P_{2n-1} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdots\frac{2n}{2n-1}$$

$$P_{2n} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdots\frac{2n}{2n+1}$$

Secundum ea, quae praemissa sunt, est

$$\frac{\pi}{2} = P_{2n-1}\cdot\frac{2n}{2n+1}\cdot\frac{2n+2}{2n+1}\cdot\frac{2n+2}{2n+3}\cdot\frac{2n+4}{2n+3}\ldots \text{ in infini.}$$

$$= P_{2n-1}\cdot F$$

$$\text{ubi } F = \frac{2n}{2n+1} \cdot \frac{2n+2}{2n+1} \cdot \frac{2n+2}{2n+3} \ldots \text{ in infinitum.}$$

Quum vero sit

$$\frac{a+2k}{a+2k+1} \cdot \frac{a+2k+2}{a+2k+1} = \frac{a^2 + 4ak + 4k^2 + 2a + 4k}{a^2 + 4ak + 4k^2 + 2a + 4k + 1}$$

$$= \frac{(a+2k+1)^2 - 1}{(a+2k+1)^2} = 1 - \frac{1}{(a+2k+1)^2}$$

etiam

$$F = \left[1 - \frac{1}{(2n+1)^2}\right] \left[1 - \frac{1}{(2n+3)^2}\right] \left[\frac{1}{(2n+5)^2}\right] \ldots$$

i. e. aequale producto cuidam e fractionibus genuinis composito esse debet.

Ergo est

$$F < 1$$

unde colligitur

$$F = 1 - f$$

poni posse, ubi f quantitas sit positiva.

Dein est

$$\frac{\pi}{2} = P_{2n} \cdot \frac{2n+2}{2n+1} \cdot \frac{2n+2}{2n+3} \cdot \frac{2n+4}{2n+3} \cdot \frac{2n+4}{2n+5} \ldots$$

$$= P_{2n} \cdot F^1$$

$$\text{ubi } F^1 = \frac{2n+2}{2n+1} \cdot \frac{2n+2}{2n+3} \cdot \frac{2n+4}{2n+3} \ldots$$

Quum vero sit

$$\frac{a+2k}{a+2k-1} \cdot \frac{a+2k}{a+2k+1} = \frac{a^2 + 4ak + 4k^2}{a^2 + 4ak + 4k^2 - 1}$$

$$= \frac{(a+2k-1)(a+2k+1)+1}{(a+2k-1)(a+2k+1)}$$

$$= 1 + \frac{1}{(a+2k-1)(a+2k+1)}$$

etiam

$$F^1 = \left[1 + \frac{1}{(2n+1)(2n+3)}\right] \left[1 + \frac{1}{(2n+3)(2n+5)}\right] \left[1 + \frac{1}{(2n+5)(2n+7)}\right] \ldots$$

i. e. aequale producto e fractionibus spuriis composito, quod per $1 + f^1$ designari potest, ubi $f^1$ quantitas est positiva, esse debet.

Ergo est

$$\frac{\Pi}{2} = P_{2n-1} \,.\, (1-f) = P_{2n}\,(1+f^{I})$$

$$= P_{2n-1} - f\,P_{2n-1} = P_{2n} + f^{I}\,P_{2n}$$

Eadem atque supra concludendi ratione adhidita erit

$$\frac{\Pi}{2} = P_{2n-1} - f\,P_{2n-1} = P_{2n} + f^{I}P_{2n} = P_{2n+1} - f^{II}P_{2n+} = P_{2n+2} + f^{III}P_{2n+2} =$$

ubi per $f$, $f^{I}$, $f^{II}$, $f^{III}$, ... quantitates positivae describuntur.

Hinc, valorem characteris $\frac{\Pi}{2}$ justum semper intra duobus valoribus partialibus se insequentibus quaerendum esse, manifestum est, seu esse

$$\frac{\Pi}{2} < P_{2n-1} \quad \text{et} \quad \frac{\Pi}{2} > P_{2n}$$

$$P_{2n-1} > \frac{\Pi}{2} > P_{2n}$$

## §. 24.

Lemma. Sit n numerus integer, tum erit

$$2n\,(2n-1)\,(2n-2)\ldots(n+2)\,(n+1) = (2n-1)\,(2n-3)\,(2n-5)\,\ldots\,5.\,3.\,1.\,2^{n}$$

et

$$(2n-1)\,(2n-2)\,(2n-3)\,\ldots\,(n+1)n = (2n-1)\,(2n-3)\,(2n-5)\,\ldots\,5.\,3.\,1.\,2^{n-1}$$

Multiplicando enim has aeqationes per

$$2\,(2n+1)$$

erit

$$(2n+2)\,(2n+1)\,2n\,(2n-1)\,\ldots\,(n+2) = (2n+1)\,(2n-1)\,(2n-3)\ldots 5.\,3.\,1.\,2^{n+1}$$

et

$$(2n+1)\,2n\,(2n-1)\,(2n-2)\,\ldots\,(n+1) = (2n+1)\,(2n-1)\,(2n-3)\ldots 5.\,3.\,1.\,2^{n}$$

Jam perspicuum est, supremo loco positam utramque aequationem ex utraque antecedenti oriri, si pro n posuerimus $n+1$, seu valere pro $n+1$ quandoquidem valeant pro n. Atque jam aequationes istae locum habent pro $n = 1, 2, 3..$, ergo secundum illam, quae a Bernullio nomen cepit, concludendi rationem in genere locum habent.

## §. 25.

Coroll. Est:

$$(2n)_n = \frac{2n\,(2n-1)\,(2n-2)\,\ldots\,(n+1)}{1.\;2.\;3.\;.\;.\;.\;.\;n}$$

$$= \frac{1.\,3.\,5\,\ldots\,(2n-3)\,(2n-1)}{2.\,4.\,6\,\ldots\ldots\,2n}\,.\,2^{2n}$$

dein

$$(2n-1)_n = \frac{(2n-1)\,(2n-2)\,\ldots\,(n+1)n}{1.\quad 2.\quad 3.\,\ldots\,n}$$

$$= \frac{1.\,3.\,5.\,7\,\ldots\,(2n-3)\,(2n-1)}{2.\,4.\,6\,\ldots\ldots\,2n}\,.\,2^{2n-1}$$

Jam vero est

$$2^a = (1+1)^a = 1 + a_1 + a_2 + a_3 + \cdots + a_1 + 1$$

i. e. aequale summae omnium unciarum potestatis $a^{tae}$, quandoquidem per a numerum positivum integrum denotes. Itaque, quoniam $(2)_n$ et $(2n-1)_n$ unciae mediae potestatis $(2n)^{tae}$ et $(2n-1)^{tae}$ sint et

$$(2n)_n : 2^{2n} = 1.\,3.\,5\,\ldots\,(2n-3)\,(2n-1) : 2.\,4.\,6\,\ldots\,2n$$

$$(2n-1)_n : 2^{2n-1} = 1.\,3.\,5\,\ldots\,(2n-3)\,(2n-1) : 2.\,4.\,6\,..\,2n$$

sit, in genere esse intelligitur: unciam mediam potestatis $a^{tae}$ ad summam unciarum omnium ejusdem potestatis ita se habere, ut productum omnium numerorum imparium ad productum omnium numerorum parium in intervallo 1 usque ad a sumtorum.

Jam vero quum esset

$$P_{2n} = \frac{2.\,2.\,4.\,4.\,6.\,6.\,8.\,8\,\ldots\,2n\,.\,2n}{1.\,3.\,3.\,5.\,5.\,7.\,7.\,9\,\ldots\,(2n-1)\,(2n+1)}$$

ergo

$$\frac{1}{P_{2n}} = \frac{1.\,3.\,3.\,5.\,5.\,7.\,7.\,9\,\ldots\,(2n-1)\,(2n+1)}{2.\,2.\,4.\,4.\,6.\,6.\,8.\,8\,\ldots\,2n\,..\,2n}$$

$$\sqrt{\frac{1}{P_{2n}}} = \frac{1}{\sqrt{P_{2n}(2n+1)}} = \frac{1.\,3.\,5.\,7\,..\,(2n-1)}{2.\,4.\,6\,..\,2n}$$

et

$$P_{2n} = P_{2n-1}\left[\frac{2n}{2n+1}\right]$$

$$P_{2n}(2n+1) = P_{2n-1}\,.\,(2n)$$

erit, unde sequitur esse

$$(2n)_n = \frac{2^{2n}}{\sqrt{P_{2n}\,(2n+1)}} = \frac{2^{2n}}{\sqrt{P_{2n-1}\,.\,2n}}$$

similique ratione

$$(2n-1)_n = \frac{2^{2n-1}}{\sqrt{P_{2n}\,(2n+1)}} = \frac{2^{2n-1}}{\sqrt{P_{2n-1}\,.\,2n}}$$

§. 25.

Jam formulae, quae inserviant computationi unciae mediae magnarum potestatum, approximativae ex iis, quae modo diximus, ita possunt accipi, ut in formulis sub finem §. 24. scriptis $\frac{\pi}{2}$ pro $P_{2n-1}$ et pro $P_{2n}$ ponatur. Tum erit

$$\frac{2^{2n}}{\sqrt{\frac{\pi}{2}(2n+1)}} = 2^{2n}\sqrt{\frac{2}{(2n+1)\pi}}$$

$$\frac{2^{2n}}{\sqrt{\frac{\pi}{2}\,.\,2n}} = 2^{2n}\sqrt{\frac{2}{2n\,\pi}} = 2^{2n}\sqrt{\frac{1}{n\,\pi}}\,.$$

Pro mediis unciis potestatum imparium eaedem formulae valent, dummodo in ipsis $2^{2n-1}$ pro $2^{2n}$ ponatur.

Istas aequationes pro magnis valoribus literae n tributis vere approximativos valores exhibere perspicuum erit, modo demonstraverimus: valores illos eo accuratius cum veris unciae mediae valoribus congruere, quo majus n acceptum fuerit. Ac jam docuimus esse

$$P_{2n-1} > \frac{\pi}{2} > P_{2n}$$

ergo

$$(2n)_n > \frac{2^{2n}}{\sqrt{\frac{\pi}{2}\,(2n+1)}} \text{ sive}$$

$$(2n)_n > 2^{2n}\sqrt{\frac{2}{(2n+1)\pi}}$$

$$\text{et} \quad (2n)_n < \frac{2^{2n}}{\sqrt{\frac{\pi}{2}\, 2n}} \quad \text{sive}$$

$$(2n)_n < 2^{2n} \sqrt{\frac{2}{2n\,\pi}}$$

i. e. valor quantitatis $(2n)_n$ media quantitas est intra

$$2^{2n} \sqrt{\frac{2}{(2n+1)\,\pi}} \quad \text{et} \quad 2^{2n} \sqrt{\frac{2}{2n\,\pi}}$$

Ergo si quantitas $(2n)_n$ sive per priorem, sive per posteriorem aequationem computatur, error quidam committitur, qui ubi maximus erit, per differentiam utriusque expressionis exarari poterit. Quem errorem si designemus litera E, semper erit

$$E < 2^{2n} \sqrt{\frac{2}{\pi}} \left[ \frac{1}{\sqrt{2n}} - \frac{1}{\sqrt{2n+1}} \right]$$

Quodsi hujus erroris rationem ad valorem justum quantitatis $(2n)_n$ per E' designemus, erit

$$E' < \frac{2^{2n} \sqrt{\frac{2}{\pi}} \left[ \frac{1}{\sqrt{2n}} - \frac{1}{\sqrt{2n+1}} \right]}{\frac{2^{2n}}{\sqrt{P_{2n} \cdot (2n+1)}}}$$

$$< \frac{2^{2n} \sqrt{\frac{2}{\pi}} \left[ \frac{1}{\sqrt{2n}} - \frac{1}{\sqrt{2n+1}} \right]}{\frac{2^{2n}}{\sqrt{\frac{\pi}{2}(2n+1)}}}$$

$$< \frac{\frac{1}{\sqrt{2n}} - \frac{1}{\sqrt{2n+1}}}{\frac{1}{\sqrt{2n+1}}}$$

$$< \frac{\sqrt{2n+1}}{\sqrt{2n}} - 1$$

$$< \sqrt{1 + \frac{1}{2n}} - 1$$

$$< \frac{1}{2} \cdot \frac{1}{2n} - \frac{1}{8}\left(\frac{1}{2n}\right)^2 + \frac{1}{16}\left(\frac{1}{2n}\right)^3 - \dots$$

Quae quantitas semper eo minor est, quo majus n acceperis, ergo etiam ratio erroris ad veram quantitatem eo minor est, quo majus n acceptum fuerit. Quum $(2n)_n$ quantitas sit, media inter

$$2^{2n}\sqrt{\frac{2}{(2n+1)\pi}} \quad \text{et} \quad 2^{2n}\sqrt{\frac{2}{2n\pi}}$$

perspicuum est, errorem eo minorem fore si ad computandum $(2n)_n$ quantitas media inter

$$2^{2n}\sqrt{\frac{2}{(2n+1)\pi}} \quad \text{et} \quad 2^{2n}\sqrt{\frac{2}{2n\pi}}$$

adhibeatur, cujusmodi quantitas est

$$2^{2n}\sqrt{\frac{2}{(2n+\frac{1}{2})\pi}} = 2^{2n}\sqrt{\frac{4}{(4n+1)\pi}} = 2^{2n+1}\sqrt{\frac{1}{(4n+1)\pi}}$$

Quarum aequationum ope quam accurate verus quantitatis nostrae valor eruatur, exemplum pro $n = 10$, demonstret. Vera unciae mediae potestatis vicesimae quantitas est 184756. Jam est

$$2^{2n}\sqrt{\frac{2}{(2n+1)\pi}} = 182570$$

$$2^{2n}\sqrt{\frac{2}{2n\pi}} = 187079$$

$$2^{2n+1}\sqrt{\frac{1}{(4n+1)\pi}} = 184773$$

Supremo loco positus valor propius accedit ad veritatem quam arithmetica ratio media priorum amborum, quae est 184825.

## §. 26.

Ponendo m, n, k pro quantitatibus positivis seu negativis quibuslibet, p pro numero positivo integro, summa seriei

$$\overset{k}{m_p} + \overset{k}{m_{p-1}} \cdot \overset{k}{n_1} + \overset{k}{m_{p-2}} \cdot \overset{k}{n_2} + \overset{k}{m_{p-3}} \cdot \overset{k}{n_3} + \dots$$

$$\dots \overset{k}{m_3} \cdot \overset{k}{n_{p-3}} + \overset{k}{m_2} \cdot \overset{k}{n_{p-2}} + \overset{k}{m_1} \cdot \overset{k}{n_{p-1}} + \overset{k}{n_p}$$

$$= F(p)$$

sit. Ac jam erit

$$\frac{m+n+pk}{p+1} = \frac{m+pk}{p+1} + \frac{n}{p+1}$$

$$= \frac{m+(p-1)k}{p+1} + \frac{n+k}{p+1}$$

$$= \frac{m+(p-2)k}{p+1} + \frac{n+2k}{p+1}$$

$$-\quad -\quad -$$

$$= \frac{m+2k}{p+1} + \frac{n+(p-2)k}{p+1}$$

$$= \frac{m+k}{p+1} + \frac{n+(p-1)k}{p+1}$$

$$= \frac{m}{p+1} + \frac{n+pk}{p+1}$$

Multiplicando seriem $F(p)$ in $\frac{m+n+pk}{p+1}$ et pro $\frac{m+n+pk}{p+1}$ ea, in quae modo dissoluta est membra, substituendo habebitur

$$F(p) \cdot \frac{m+n+pk}{p+1} = \overset{k}{m_p} \cdot \frac{m+pk}{p+1} + \overset{k}{m_p} \cdot \frac{n}{p+1}$$

$$+ \overset{k}{m_{p-1}} \cdot \frac{m+(p-1)k}{p+1} \cdot \overset{k}{n_1} + \overset{k}{m_{p-1}} \cdot \frac{n+k}{p+1} \cdot \overset{k}{n_1}$$

$$+ \overset{k}{m_{p-2}} \cdot \frac{m+(p-2)k}{p+1} \cdot \overset{k}{n_2} + \overset{k}{m_{p-2}} \cdot \frac{n+2k}{p+1} \cdot \overset{k}{n_2}$$

$$+ \overset{k}{m_{p-3}} \cdot \frac{m+(p-3)k}{p+1} \cdot \overset{k}{n_3} + \overset{k}{m_{p-3}} \cdot \frac{n+3k}{p+1} \cdot \overset{k}{n_3}$$

$$-\quad -\quad -$$

$$+ \overset{k}{m_2} \cdot \frac{m+2k}{p+1} \cdot \overset{k}{n_{p-2}} + \overset{k}{m_2} \cdot \frac{n+(p-2)k}{p+1} \cdot \overset{k}{n_{p-2}}$$

$$+ \overset{k}{m_1} \cdot \frac{m+k}{p+1} \cdot \overset{k}{n_{p-1}} + \overset{k}{m_1} \cdot \frac{n+(p-1)k}{p+1} \cdot \overset{k}{n_{p-1}}$$

$$+ \frac{m}{p+1} \cdot \overset{k}{n_p} + \frac{n+pk}{p+1} \cdot \overset{k}{n_p}$$

Quum autem sit

$$\overset{k}{m}_p \cdot \frac{m + pk}{p + 1} = \overset{k}{m}_p \cdot \frac{m + pk}{p + 1} \cdot \frac{p + 1}{p + 1} = \overset{k}{m}_{p+1} \cdot \frac{p + 1}{p + 1}$$

$$\overset{k}{m}_{p-1} \cdot \frac{m + (p-1)k}{p + 1} = \overset{k}{m}_{p-1} \cdot \frac{m + (p-1)k}{p} \cdot \frac{p}{p + 1} = \overset{k}{m}_p \cdot \frac{p}{p + 1}$$

$$\overset{k}{m}_{p-2} \cdot \frac{m + (p-2)k}{p + 1} = \overset{k}{m}_{p-2} \cdot \frac{m + (p-2)k}{p-1} \cdot \frac{p-1}{p + 1} = \overset{k}{m}_{p-1} \cdot \frac{p-1}{p + 1}$$

$$\overset{k}{m}_{p-3} \cdot \frac{m + (p-3)k}{p + 1} = \overset{k}{m}_{p-3} \cdot \frac{m + (p-3)k}{p-2} \cdot \frac{p-2}{p + 1} = \overset{k}{m}_{p-2} \cdot \frac{p-2}{p + 1}$$

— — — — — — —

$$\overset{k}{m}_2 \cdot \frac{m + 2k}{p + 1} = \overset{k}{m}_2 \cdot \frac{m + 2k}{3} \cdot \frac{3}{p + 1} = \overset{k}{m}_3 \cdot \frac{3}{p + 1}$$

$$\overset{k}{m}_1 \cdot \frac{m + k}{p + 1} = \overset{k}{m}_1 \cdot \frac{m + k}{2} \cdot \frac{2}{p + 1} = \overset{k}{m}_2 \cdot \frac{2}{p + 1}$$

$$\frac{m}{p + 1} = \frac{m}{1} \cdot \frac{1}{p + 1}$$

eundemque ad modum

$$\frac{n}{p + 1} = \frac{n}{1} \cdot \frac{1}{p + 1} = \overset{k}{n}_1 \cdot \frac{1}{p + 1}$$

$$\overset{k}{n}_1 \cdot \frac{n + k}{p + 1} = \overset{k}{n}_1 \cdot \frac{n + k}{2} \cdot \frac{2}{p + 1} = \overset{k}{n}_2 \cdot \frac{2}{p + 1}$$

$$\overset{k}{n}_2 \cdot \frac{n + 2k}{p + 1} = \overset{k}{n}_2 \cdot \frac{n + 2k}{3} \cdot \frac{3}{p + 1} = \overset{k}{n}_3 \cdot \frac{3}{p + 1}$$

$$\overset{k}{n}_3 \cdot \frac{n + 3k}{p + 1} = \overset{k}{n}_3 \cdot \frac{n + 3k}{4} \cdot \frac{4}{p + 1} = \overset{k}{n}_4 \cdot \frac{4}{p + 1}$$

— — — — —

$$\overset{k}{n}_{p-2} \cdot \frac{n + (p-2)k}{p + 1} = \overset{k}{n}_{p-2} \cdot \frac{n + (p-2k)}{p-1} \cdot \frac{p-1}{p + 1} = \overset{k}{n}_{p-1} \cdot \frac{p-1}{p + 1}$$

$$\overset{k}{n}_{p-1} \cdot \frac{n + (p-1)k}{p + 1} = \overset{k}{n}_{p-1} \cdot \frac{n + (p-1)k}{p} \cdot \frac{p}{p + 1} = \overset{k}{n}_p \cdot \frac{p}{p + 1}$$

$$\overset{k}{n}_{p} \cdot \frac{n+pk}{p+1} = \overset{k}{n}_{p} \cdot \frac{n+pk}{p+1} \cdot \frac{p+1}{p+1} = \overset{k}{n}_{p+1} \cdot \frac{p+1}{p+1}$$

ergo etiam est

$$\begin{aligned}
F(p) \cdot \frac{m+n+pk}{p+1} = {} & \overset{k}{m}_{p+1} \cdot \frac{p+1}{p+1} + \overset{k}{m}_{p} \cdot \overset{k}{n}_{1} \cdot \frac{1}{p+1} \\
& + \overset{k}{m}_{p} \cdot \overset{k}{n}_{1} \cdot \frac{p}{p+1} + \overset{k}{m}_{p-1} \cdot \overset{k}{n}_{2} \cdot \frac{2}{p+1} \\
& + \overset{k}{m}_{p-1} \cdot \overset{k}{n}_{2} \cdot \frac{p+1}{p+1} + \overset{k}{m}_{p-2} \cdot \overset{k}{n}_{3} \cdot \frac{3}{p+1} \\
& + \overset{k}{m}_{p-2} \cdot \overset{k}{n}_{3} \cdot \frac{p-2}{p+1} + \overset{k}{m}_{p-3} \cdot \overset{k}{n}_{4} \cdot \frac{4}{p+1} \\
& \qquad - \qquad - \qquad - \qquad - \qquad - \\
& + \overset{k}{m}_{3} \cdot \overset{k}{n}_{p-2} \cdot \frac{3}{p+1} + \overset{k}{m}_{2} \cdot \overset{k}{n}_{p-1} \cdot \frac{p-1}{p+1} \\
& + \overset{k}{m}_{2} \cdot \overset{k}{n}_{p-1} \cdot \frac{2}{p+1} + \overset{k}{m}_{1} \cdot \overset{k}{n}_{p} \cdot \frac{p}{p+1} \\
& + \overset{k}{m}_{1} \cdot \overset{k}{n}_{p} \cdot \frac{1}{p+1} + \overset{k}{n}_{p+1} \cdot \frac{p+1}{p+1} \\
= {} & \overset{k}{m}_{p+1} \cdot \frac{p+1}{p+1} \\
& + \overset{k}{m}_{p} \cdot \overset{k}{n}_{1} \left[\frac{p}{p+1} + \frac{1}{p+1}\right] \\
& + \overset{k}{m}_{p-1} \cdot \overset{k}{n}_{2} \left[\frac{p-1}{p+1} + \frac{2}{p+1}\right] \\
& + \overset{k}{m}_{p-2} \cdot \overset{k}{n}_{3} \left[\frac{p-2}{p+1} + \frac{3}{p+1}\right] \\
& + \overset{k}{m}_{p-3} \cdot \overset{k}{n}_{4} \left(\frac{p-3}{p+1} + \frac{4}{p+1}\right) \\
& \qquad - \qquad - \qquad - \qquad -
\end{aligned}$$

$$+ \overset{k}{m}_3 \cdot \overset{k}{n}_{p-2} \left( \frac{3}{p+1} + \frac{p-2}{p+1} \right)$$

$$+ \overset{k}{m}_2 \cdot \overset{k}{n}_{p-1} \left( \frac{2}{p+1} + \frac{p-1}{p+1} \right)$$

$$+ \overset{k}{m}_1 \cdot \overset{k}{n}_p \cdot \left( \frac{1}{p+1} + \frac{p}{p+1} \right)$$

$$+ \overset{k}{n}_{p+1} \cdot \frac{p+1}{p+1}$$

$$= \overset{k}{m}_{p+1} + \overset{k}{m}_p \cdot \overset{k}{n}_1 + \overset{k}{m}_{p-1} \cdot \overset{k}{n}_2 + \overset{k}{m}_{p-2} \cdot \overset{k}{n}_3 + \overset{k}{m}_{p-3} \cdot \overset{k}{n}_4 + \cdots$$

$$\cdots + \overset{k}{m}_3 \cdot \overset{k}{n}_{p-2} + \overset{k}{m}_2 \cdot \overset{k}{n}_{p-1} + \overset{k}{m}_1 \cdot \overset{k}{n}_p + \overset{k}{n}_{p+1}$$

Sicuti vero

$$F(p) = \overset{k}{m}_p + \overset{k}{m}_{p-1} \cdot \overset{k}{n}_1 + \overset{k}{m}_{p-2} \cdot \overset{k}{n}_2 + \cdots$$

$$\cdots \overset{k}{m}_3 \cdot \overset{k}{n}_{p-3} + \overset{k}{m}_2 \cdot \overset{k}{n}_{p-2} + \overset{k}{m}_1 \cdot \overset{k}{n}_{p-1} + \overset{k}{n}_p$$

posuimus, ita ex aequationis natura

$$F(p+1) = \overset{k}{m}_{p+1} + \overset{k}{m}_p \cdot \overset{k}{n}_1 + \overset{k}{m}_{p-1} \cdot \overset{k}{n}_2 + \overset{k}{m}_{p-2} \cdot \overset{k}{n}_3 + \cdots$$

$$\cdots \overset{k}{m}_3 \cdot \overset{k}{n}_{p-2} + \overset{k}{m}_2 \cdot \overset{k}{n}_{p-1} + \overset{k}{m}_1 \cdot \overset{k}{n}_p + \overset{k}{n}_{p+1}$$

ponere nos licebit. Ergo ex iis, quae modo exposita sunt, haec insignis aequatio habebitur:

$$F(p+1) = F(p) \cdot \frac{m+n+pk}{p+1}$$

Jam vero quum sit

$$F(1) = \overset{k}{m}_1 + \overset{k}{n}_1 = \frac{m+n}{1}$$

secundum aequationem modo laudatam est

$$F(2) = F(1) \cdot \frac{m+n+k}{2} \left( = \frac{m+n}{1} \cdot \frac{m+n+k}{2} \right)$$

$$F(3) = F(2) \cdot \frac{m+n+2k}{3} \left( = \frac{m+n}{1} \cdot \frac{m+n+k}{2} \cdot \frac{m+n+2k}{3} \right)$$

$$F(4) = F(3) \cdot \frac{m+n+3k}{4} \left( = \frac{m+n}{1} \cdot \frac{m+n+k}{2} \cdot \frac{m+n+2k}{3} \cdot \frac{m+n+3k}{4} \right)$$

Qua ratione quomodo sit progrediendum neminem fugit, quamque legem istae expressiones sequantur, in liquido constat. Est enim

$$F(p) = \frac{(m+n)\,(m+n+k)\,(m+n+2k)\,\ldots\,[m+n+(p-1)k]}{1.\;2.\;3\;.\;.\;.\;.\;p}$$

i. e.

$$F(p) = \overset{k}{(m+n)}_p$$

Hinc igitur accipitur haec relatio notabilis

a) $$\overset{k}{(m+n)}_p = \overset{k}{m}_p + \overset{k}{m}_{p-1} \cdot \overset{k}{n}_1 + \overset{k}{m}_{p-2} \cdot \overset{k}{n}_2 + \overset{k}{m}_{p-3} \cdot \overset{k}{n}_3 + \cdots$$

$$\ldots \overset{k}{m}_3 \cdot \overset{k}{n}_{p-3} + \overset{k}{m}_2 \cdot \overset{k}{n}_{p-2} + \overset{k}{m}_1 \cdot \overset{k}{n}_{p-1} + \overset{k}{n}_p$$

unde, $k = -1$ posito, prodibit aequatio

b) $$(m+n)_p = m_p + m_{p-1} \cdot n_1 + m_{p-2} \cdot n_2 + m_{p-2} \cdot n_3 + \cdots$$

$$\ldots m_3 \cdot n_{p-3} + m_2 \cdot n_{p-2} + m_1 \cdot n_{p-1} + n_p$$

Theorema b) ea de causa gravissimum est, quod ipso nititur demonstratio universalis simplicissima **theorematis binomialis**; m et n quilibet positivi aut negativi, integri aut fracti numeri esse possunt, quoniam formulae, unde theorema illud derivatum est, locum habent pro omnibus numerorum speciebus. Demonstratio ejusdem theorematis speciosa, quamvis non sit tam generalis ut modo exhibita, legitur in libro:

Supplemente zum math. Lexicon Klügels, composito ab ill. **Grunerto**, praeceptore summopere mihi colendo.

Tertia demonstratio invenitur in libro cui inscriptum est: math. Lex. v. Klügel Art. Binomial-Coefficienten.

## §. 27.

Pro p si posueris successive valores 1, 2, 3, 4 . . ., habebis aequationes has

$$(m+n)_1 = 1 \,.\, m_1 + n_1 \,.\, 1$$

$$(m+n)_2 = 1 \,.\, m_2 + n_1 \,.\, m_1 + n_2 \,.\, 1$$

$$(m+n)_3 = 1 \,.\, m_3 + n_1 \,.\, m_2 + n_2 \,.\, m_1 + n_3 \,.\, 1$$

$$(m+n)_4 = 1 \,.\, m_4 + n_1 \,.\, m_3 + n_2 \,.\, m_2 + n_3 \,.\, m_1 + n_4 \,.\, 1$$

## §. 28.

Theoremate modo demonstrato auxiliante producta duarum, trium et plurium serierum inveniri possunt, modo series ex unciis cum potestatibus indefinitae quantitatis z conjunctis compositae sint. Multiplicemus primo ambas sequentes series alteram cum altera:

$$M = 1 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \ldots$$

$$N = 1 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 + \ldots$$

Sit $M \,.\, N = P$

$$= 1 + az + bz^2 + cz^3 + dz^4 + \ldots$$

Secundum §. 27. aequatur

$$a = (a+b)_1 ; \quad b = (a+b)_2 ; \quad c = (a+b)_3, \ldots$$

ita ut sit $M \,.\, N = P$

$$= 1 + (a+b)_1 z + (a+b)_2 z^2 + (a+b) z^3 + \ldots$$

Productum serierum M et N ejusdem est formae, cum seriebus M et N ipsis h. e. unciae potestatum numeri z ex $a + b$ eadem ratione accipiuntur atque unciae in serie M positae ex $a$ et unciae in serie N positae e $b$, ubi et $a$ et $b$ numeri positivi aut negativi, integri aut fracti esse possunt.

Atque jam significemus series

1) $1 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \ldots$

2) $1 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 + \ldots$

3) $1 + g_1 z + g_2 z^2 + g_3 z^3 + g_4 z^4 + \ldots$

m) $1 + m_1 z + m_2 z^2 + m_3 z^3 + m_4 z^4 + \ldots$

respective per

$$ {}^{a}S, \quad {}^{b}S, \quad {}^{g}S, \quad \ldots \quad {}^{m}S, $$

productum istarum serierum omnium eadem cum singulis seriebus gaudere debet forma. Ergo si

$$ a + b + g + \ldots + m = n $$

ponatur, erit

1.) $\quad {}^{a}S \cdot {}^{b}S \cdot {}^{g}S \cdot \ldots \cdot {}^{m}S = {}^{n}S$

Pro $a = b = g = \ldots = m$ erit $n = m\,a$

et

2.) $\quad [{}^{a}S]^{m} = {}^{m\,a}S$

Ergo est, si $a = \frac{n}{m}$ ponatur

3.) $\quad {}^{a}S = [{}^{n}S]^{\frac{1}{m}}$

(ubi a numerum fractum aut integrum denotare potest, quia formulae unde hoc theorema derivatum est, pro integris et fractis a valent) seu

$$ \left[{}^{n}S\right]^{\frac{1}{m}} = 1 + \left[\frac{n}{m}\right]_1 z + \left(\frac{n}{m}\right)_2 z^2 + \left(\frac{n}{m}\right)_3 z^3 + \ldots $$

unde perspicuum est, exponentem unciarum fractum $\frac{n}{m}$ extractionem radicis m^tae^ e serie

4.) $\quad 1 + n_1 z + n_2 z^2 + n_3 z^3 + n_4 z^4 + \ldots$

denotare.

Exponens **negativus** seriei ${}^{-n}S$ divisionem unitatis per ${}^{+n}S$ significat. Sit enim $m > n$, erit

5.) $\quad {}^{m}S = {}^{n}S \cdot {}^{m-n}S$

ergo

$$ \frac{{}^{m}S}{{}^{n}S} = {}^{m-n}S $$

Quotientium forma etiam pro $m < n$ immutata manet, ergo etiam pro exponente negativo eadem est cum seriebus ${}^mS$ et ${}^nS$ in casu $m < n$. Pro $m = 0$ est

$$ {}^0S = 1 $$

$$ \text{et} \quad \frac{1}{{}^nS} = {}^{-n}S $$

$$ \text{i. e.} \quad \frac{1}{{}^nS} = 1 + (-n_1)\,z + (-n)_2\,z^2 + (-n)_3\,z^3 + \ldots $$

cujus relationis coëfficientes accipiunt valores e. 4.) modo pro n positum erit — n.

## §. 29.

Ponendo in serie §. 26. b.)

$$ 1 \,.\, b_n + a_1\, b_{n-1} + a_2\, b_{n-2} + \ldots + a_n \cdot 1 = (a+b)_n $$

$$ b = n $$

habebitur aeqatio clarissima haec:

$$ 1 \,.\, n_n + a_1\, n_{n-1} + a_2\, n_{n-2} + \ldots + a_{n-1}\, n_1 + a_n \cdot 1 $$

$$ = 1 \,.\, 1 + a_1\, n_1 + a_2\, n_2 + \ldots + a_{n-1}\, n_1 + a_n \cdot 1 $$

$$ = (a+n)_n $$

## §. 30.

Theorema in §. 29. enunciatum e formulis in §. 16. explicatis hac ratione derivari potest. Docuimus esse

1.) $(n+1)_{m+1} = n_m + n_{m+1}$

2.) $\quad = (n-1)_{m-1} + 2\,(n-1)_m + (n-1)_{m+1}$

3.) $\quad = (n-2)_{m-2} + 3\,(n-2)_{m-1} + 3\,(n-2)_m + (n-2)_{m+1}$

4.) $\quad = (n-3)_{m-3} + 4\,(n-3)_{m-2} + 6\,(n-3)_{m-1} + 4\,(n-3)_m + (n-3)_{m+1}$

$+ \quad - \quad - \quad - \quad - \quad -$

Si in aequatione 4.) ponamus $n - 3 = p$, et $m - 3 = q$, erit $n + 1 = p + 4$, $m + 1 = q + 4$, $m = q + 3$, $m - 1 = q + 2$; $m - 2 = q + 1$; ergo erit

5.) $(p+4)_{q+4} = p_q + 4\,p_{q+1} + 6p_{q+2} + 4\,p_{q+3} + p_{q+4}$

Jam si possit demonstrari, relationem modo dictam valere pro $(p+n+1)_{q+n+1}$, si valeat pro $(p+n)_{q+n}$, tum secundum illam demonstrandi rationem, quae Bernullii nomen habet, in genere veram esse perspicuum esset. Sit igitur

6.) $(p+n)_{q+n} = p_q + n_1\,p_{q+1} + n_2\,p_{q+2} + \cdot\cdot + n_1\;p_{q+n-1} + p_{q+n}$

Secundum §. 9. est

$$p_q = (p-1)_{q-1} + (p-1)_q$$

$$n_1\,p_{q+1} = n_1\,(p-1)_q + n_1\,(p-1)_{q+1}$$

$$n_2\,p_{q+2} = n_2\,(p-1)_{q+1} + n_2\,(p-1)_{q+2}$$

— — — —

$$n_1\,p_{q+n-1} = n_1\,(p-1)_{q+n-2} + n_1\,(p-1)_{q+n-1}$$

$$p_{q+n} = (p-1)_{q+n-1} + (p-1)_{q+n}$$

Unde sequitur

7.) $(p+n)_{q+n} = (p-1)_{q-1} + (1+n_1)\,(p-1)_q + (n_1+n_2)\,(p-1)_{q+1} + \cdot\cdot$

$$\cdot\;\cdot\;\cdot\;(n_1 + 1)\,(p-1)_{q+n-1} + (p-1)_{q+n}$$

Scribendo pro $p-1$ et pro $q-1$ resp. $p$ et $q$, erit, quoniam

$$1 + n_1 = (n+1)_1\;;\;\;(n_1+n_2) = (n+1)_2\;;\;\;(n_2+n_3) = (n+1)_3 \cdots$$

$$\cdot\;\cdot\;\cdot\;n_1 + 1 = (n+1)_1$$

est, habebitur

$$(p+n+1)_{q+n+1} = p_q + (n+1)_1\;p_{q+1} + (n+1)_2\;p_{q+2} + \cdots$$

$$\cdot\;\cdot\;\cdot + (n+1)_1\;\;p_{q+n} + p_{q+n+1}$$

quae relatio eandem habet formam cum aequatione 6.) atque e 6.) accipitur ponendo $n+1$ pro $n$. Ergo theorema verum est. Scribendo $q=o$, $n$ pro $n+1$ erit

$$(p+n)_n = 1\,.\,1 + n_1\,p_1 + n_2\,p_2 + \cdots + n_1\,p_{n-1} + p_n$$

Ejusdem gravissimae aequationis plures inveniuntur demonstrationes *) e quibus unam hoc loco ea de causa adumbremus, quia numeros figuratos alia, ut in §. 15., docuimus methodo nisi, explicare volumus. Demonstratio, quam hic proponere nobis est in mente, a Buzengeigero inventa est.

Secundum praecepta calculi differentialis est

1.) $\triangle^a_y = y - a_1 y^{(1)} + a_2 y^{(2)} - a_3 y^{(3)} + \cdot\cdot + (-1)^{a-1} a_1 y^{(a-1)} + (-1)^a y^{(a)}$

2.) $y^{(a)} = y + a_1 \triangle y + a_2 \triangle^2_y + a_3 \triangle^3 y + \cdot\cdot + a_1 \triangle^{a-1} y + \triangle^a y$ **)

ubi

$$y = F(x)$$
$$y^{(1)} = F(x + \mathfrak{d}x)$$
$$y^{(2)} = F(x + 2\mathfrak{d}x)$$
$$- \quad - \quad -$$
$$y^{(a)} = F(x + a\mathfrak{d}x)$$

est. Jam si posuerimus pro $y,\ y^{(1)},\ y^{(2)},\ y^{(3)}, \cdot\cdot$

$$\ldots y^{(a)} \text{ resp. } 1,\ \frac{b_1}{g_1},\ \frac{b_2}{g_2},\ \frac{b_3}{g_3},\ \cdot\cdot \frac{b_a}{g_a}$$

ex aequatione 1.) habebimus relationem hanc:

3.) $\mathfrak{d}^a y = 1 - \frac{a_1 b_1}{g_1} + \frac{a_2 b_2}{g_2} - \frac{a_3 b_3}{g_3} + \cdot\cdot + (-1)^{a-1} \frac{a_1 b_{a-1}}{g_{a-1}} + (-1)^a \frac{b_a}{g_a}$

Ac jam est

a.) $y^{(1)} - y = \mathfrak{d}y$

b) $y^{(2)} - y^{(1)} = \mathfrak{d}y^{(1)}$; $\mathfrak{d}y^{(1)} - \mathfrak{d}y = \mathfrak{d}^2 y$

Nam $y^{(2)} - y^{(1)} = y + 2\mathfrak{d}y + \mathfrak{d}^2 y - (y + \mathfrak{d}y) = \mathfrak{d}y - \mathfrak{d}^2 y = \mathfrak{d}y^{(1)}$

unde sequitur $\mathfrak{d}y^{(1)} - \mathfrak{d}y = \mathfrak{d}^2 y$

c.) $y^{(3)} - y^{(2)} = y + 3\mathfrak{d}y + 3\mathfrak{d}^2 y + \mathfrak{d}^3 y - (y + \mathfrak{d}y + \mathfrak{d}^2 y) = \mathfrak{d}y + 2\mathfrak{d}^2 y + \mathfrak{d}^3 y = \mathfrak{d}y^{(2)}$

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*) Archiv der reinen und angewandten Mathematik von Hindenburg. Bd. 2. P. 161: Von einigen merkwürdigen Eigenschaften der Binomial-Coëfficienten.
Klügels math. Lexikon: Artikel: Binomial-Coëfficienten.
Annales de Mathématiques pures et appliquées redig. par Gergonne T. 16. Ur. VIII. Ferrier 1836.

**) Quum signum differentiale $\triangle$ in talibus formulis vulgo usitatum typographo deesset, illius loco ubique signum germanicum $\mathfrak{d}$ positum est; quod mihi lector benevolus excusare velit. Auctor.

unde est

$$\mathfrak{d}y^{(2)} - \mathfrak{d}y^{(1)} = \mathfrak{d}y + 2\mathfrak{d}y + \mathfrak{d}^3 y - (\mathfrak{d}y + \mathfrak{d}^2 y) = \mathfrak{d}^2 y + \mathfrak{d}^3 y = \mathfrak{d}^2 y^{(1)}$$

et $\mathfrak{d}^2 y^{(1)} - \mathfrak{d}^2 y = \mathfrak{d}^3 y$

Quomodo simili ratione ulterius progredi liceat, per se patet. Secundum ea, quibus uti constituimus signis erit

$$\mathfrak{d}y = \frac{b}{g} - 1 = \frac{b-g}{1}$$

$$\mathfrak{d}y^{(1)} = \frac{b_2}{g_2} - \frac{b_1}{g_1} = \frac{b(b-g)}{g(g-1)}\,; \quad \mathfrak{d}^2 y = \frac{b(b-g)}{g(g-1)} - \frac{b-g}{g} = \frac{(b-g)(b-g+1)}{g(g-1)}$$

$$\mathfrak{d}y^{(2)} = \frac{b_3}{g_3} - \frac{b_2}{g_2} = \frac{b(b-1)(b-g)}{g(g-1)(g-2)}\,; \quad \mathfrak{d}^2 y^{(1)} = \frac{b(b-1)(b-g)}{g(g-1)(g-2)} - \frac{b(b-g)}{g(g-1)} = \frac{b(b-g)(b-g+1)}{g(g-1)(g-2)}\,;$$

$$\mathfrak{d}^3 y = \frac{b(b-g)(b-g+1)}{g(g-1)(g-2)} - \frac{(b-g)(b-g+1)}{g(g-1)} = \frac{(b-g)(b-g+1)(b-g+2)}{g(g-1)(g-2}$$

— — — — —

Unde consequitur esse

$$\mathfrak{d}y = \frac{b_1 - g_1}{g_1}$$

$$\mathfrak{d}^2 y = \frac{(b-g+1)_2}{g_2}$$

$$\mathfrak{d}^3 y = \frac{(b-g+2)_3}{g_3}$$

$$\mathfrak{d}^4 y = \frac{(b-g+3)_4}{g_4}$$

— — —

Quam legem istae differentiae sequantur perspicuum est, ergo in genere est

4.) $$\mathfrak{d}^a y = \frac{(a+b-g-1)_a}{g_a}$$

Hinc facile obtinentur relationes sequentes:

5.) $$1 - \frac{a_1 b_1}{g_1} + \frac{a_2 b_2}{g_2} - \frac{a_3 b_3}{g_3} + \,..\, + (-1)^{a-1}\frac{a_1 b_{a-1}}{g_{a-1}} + (-1)^a \cdot \frac{1 \cdot b_a}{g_a}$$

$$= (-1)^a \cdot \frac{(a+b-g-1)_a}{g_a}$$

6.) $$1 + \frac{a_1(b-g)_1}{g_1} + \frac{a_2(b-g+1)_2}{g_2} + \ldots + \frac{1.(a+b-g-1)_a}{g_a} = \frac{b_a}{g_a}$$

Quodsi ponamus in 5.) $g = -1$, erit

$$g_1 = -1;\ g_2 = 1;\ g_3 = -1;\ g_4 = 1, \ldots$$

7. $$1.1 + a_1 b_1 + a_2 b_2 + a_3 b_3 + \ldots + 1.b_a = (a+b)_a$$

Ponendo in 5.) $b = -1$ erit

8.) $$1 + \frac{a_1}{g_1} + \frac{a_2}{g_2} + \frac{a_3}{g_3} + \ldots + \frac{1}{g_a} = \frac{g+1}{g+1-a}$$

quoniam $(-1)^a . \frac{(a-g-2)_a}{g_a} = \frac{g+1}{g+1-a}$ sit

Multiplicando utramque aeqationis 8.) partem per $g_a$ erit

9.) $$g_a + \frac{a_1 g_a}{g_1} + \frac{a_2 g_a}{g_2} + \frac{a_3 g_a}{g_3} + \ldots + \frac{a_1 g_a}{g_{a-1}} + \frac{1.g_a}{g_a} = \frac{g_a(g+1)}{g+1-a}$$

Quum vero in genere sit

$$a_m \cdot \frac{b_a}{b_{a-m}} = a_m \cdot \frac{\frac{b(b-1)\ldots(b-a+m+2)(b-a+m+1)\ldots(b-a+1)}{1.\ 2.\ \ldots(a-m+1)\ (a-m)\ \ldots\ (a-1)a}}{\frac{b(b-1)\ \ldots\ldots\ (b-a+m+1)}{1.2\ \ldots\ldots\ (a-m)}}$$

$$= \frac{a(a-1)(a-2)\ldots(a-m+1)}{1.\ 2.\ 3.\ \ldots\ m} \cdot \frac{(b-a+m)(b-a+m-1)\ldots(b-a+1)}{(a-m+1)(a-m+2)\ldots(a-1)a}$$

$$= \frac{(b-a+m)(b-a+m-1)\ldots(b-a+1)}{1.\ 2.\ 3.\ 4\ \ldots\ m}$$

$$= (b-a+m)_m$$

ergo etiam

10.) $$\frac{g_a(g+1)}{g+1-a} = g_a + (g-1)_{a-1} + (g-2)_{a-2} + (g-3)_{a-3} + \ldots + (g-a+1)_1 + (g-a)_0$$

esse debet. In qua relationesi $g = a + 1$, ergo $a = b - 1$ ponatur habebitur

$$\frac{g_{g-1}(g+1)}{g+1-(g-1)} = \frac{g_1(g+1)}{2} = \frac{g(g+1)}{1.2}$$

$$= g_1 + (g-1)_1 + (g-2)_1 + \ldots + 4_1 + 3_1 + 2_1 + 1_1$$

$$= g + g-1 + g-2 + \ldots + 4 + 3 + 2 + 1$$

Posito in 10.) $g = a + 2$, ergo $a = g - 2$ erit

$$\frac{g_{g-2}(g+1)}{3} = \frac{g_2(g+1)}{3} = \frac{g(g-1)}{1.\ 2.}\frac{(g+1)}{2} = \frac{(g-1)g\ (g+1)}{1.\quad 2.\quad 3}$$

$$= g_{g-2} + (g-1)_{g-3} + (g-2)_{g-4} + \ldots + (g-[g-2]+1)_1 + (g-[g-2])_0$$

$$= g_2 + (g-1)_2 + (g-2)_2 + \ldots + 4_2 + 3_2 + 2_2$$

quoniam in genere sit

$$(n-a)_{n-a-2} = \frac{(n-a)\ (n-a-1)\ (n-a-2) \ldots (n-a-[n-a-2]+2)\ (n-a-[n-a-2]+1)}{1.\quad 2.\quad 3.\quad 4.\quad 5.\ .\ .\ (n-a-3)\ (n-a-2)}$$

$$= \frac{(n-a)\ (n-a-1)\ (n-a-2)\ldots 6.\ 5.\ 4.\ 3}{1.\ 2.\ 3.\ 4 \ldots (n-a-3)\ (n-a-2)}$$

$$= \frac{(n-a)\ (n-a-1)}{1.\quad 2}$$

$$= (n-a)_2$$

Ergo est

$$\frac{(g-1)g\ (g+1)}{1.\quad 2.\quad 3.} = \frac{2\ .\ 1}{1\ .\ 2} + \frac{3\ .\ 2}{1\ .\ 2} + \frac{4\ .\ 3}{1\ .\ 2} + \frac{5\ .\ 4}{1\ .\ 2} + \ldots + \frac{(g-1)\ (g-2)}{1.\quad 2} + \frac{g\,(g-1)}{1.\quad 2}$$

Hac ratione ceteros numeros figuratos accipi posse nemo vocabit in dubium cf. §. 15.

## §. 31.

Ponendo in aequatione

$$(a+g)_n = 1\,.\,a_n + a_{n-1}g_1 + a_{n-2}g_2 + \ldots + a_2 g_{n-2} + a_1 g_{n-1} + g_n$$

$$a = g = n$$

habebitur

$$1^2 + (a_1)^2 + (a_2)^2 + (a_3)^2 + \ldots + (a_1)^2 + 1^2 = (2a)_a$$

$$= \frac{1\ .\ 3\ .\ 5 \ldots (2a-1)}{2\ .\ 4\ .\ 6 \ldots\ 2a}\,.\,2^{2a}$$