Ut varias fractionum continuarum disquisitiones hujus libelli argumentum eligerem, studio et amore potissimum doctrinae numerorum impulsus esse mihi videor, quippe quae arcte cum illa doctrina cohaereat. Quantam autem voluptatem hujus arithmeticae sublimioris partis studium afferat, nemo non ignoret, qui in ea non mediocriter versatus est. Quum igitur abhinc tres annos doctrinae fractionum continuarum me dederim novaque quaedam attentione, ut puto, non plane indigna repererim, vigiliarum jam fructus in lucem edere constitui.

Distribui totam materiam in sectiones quinque, quae ab se in vicem non pendent, quarumque ultimas duas gravissimas puto. Quod sit earum cujusque argumentum, id ipsum opus te edocebit.

Sectio I.

Connexus fractionum convergentium, quae in fractione continua simplici evolutione radicis $\sqrt{A}$ exorta eidem periodorum quotienti completo respondeant.

Satis notum est, evolutione radicis $\sqrt{A}$, designante A numerum quencunque integrum, fractionem periodicam hujus formae oriri

$$\sqrt{A} = a + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{2a + \cfrac{1}{a_1 + \text{etc.}}}}}$$

ita ut periodus a quotiente secundo incipiens quotiente, qui est duplum numeri maximi integri a, radice $\sqrt{A}$ comprehensi, terminetur. Designemus fractiones convergentes per $\frac{p}{q}, \frac{p_1}{q_1}, \frac{p_2}{q_2}$, etc. atque multitudinem periodi terminorum per k.

Maximi hic momenti sunt fractiones $\frac{p_{k-1}}{q_{k-1}}, \frac{p_{2k-1}}{q_{2k-1}}, \frac{p_{3k-1}}{q_{3k-1}}$, etc., paenultimo cujusque periodi quotienti respondentes, quarum quaelibet $\frac{p_{mk-1}}{q_{mk-1}}$, in m^{ta} periodo posita, aequationi satisfaciat

$$x^2 - Ay^2 = (-1)^{mk}.$$

Vice versa quum, si integri x, y aequationi satisfaciant $x^2 - Ay^2 = \pm 1$, fractio $\frac{x}{y}$ inter fractiones convergentes radicis $\sqrt{A}$ occurrere debeat *), hujus ipsius aequationis solutio a sola evolutione radicis $\sqrt{A}$ in fractionem continuam simplicem pendebit.

Quando igitur k est numerus par, aequatio quidem $x^2 - Ay^2 = -1$ resolvi nequit, haec vero $x^2 - Ay^2 = +1$ fractionibus convergentibus omnium periodorum resolvitur.

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*) Legendre Théorie des nombres. Trois. édit. Paris 1830., p. 25. sqq.

Sin k est numerus impar, fractiones convergentes periodorum imparis ordinis aequationi primae, secundae vero fractiones periodorum paris ordinis satisfacient.

Itaque ut omnes aequationis $x^2 - Ay^2 = \pm 1$ radices inveniantur, functionem quaeri oportet inter fractiones $\frac{p_{k-1}}{q_{k-1}}, \frac{p_{mk-1}}{q_{mk-1}}$, quarum prima minimas aequationis commemoratae radices comprehendit. Cujus solvendi problematis Ill. Legendrius *) hac fere methodo usus est:

Quoniam numeri aequationibus determinati

$$x + y\sqrt{A} = (p_{k-1} + q_{k-1}\sqrt{A})^m$$
$$x - y\sqrt{A} = (p_{k-1} - q_{k-1}\sqrt{A})^m,$$

qui sunt

$$(a)\quad \begin{cases} x = \dfrac{(p_{k-1} + q_{k-1}\sqrt{A})^m + (p_{k-1} - q_{k-1}\sqrt{A})^m}{2} \\ y = \dfrac{(p_{k-1} + q_{k-1}\sqrt{A})^m - (p_{k-1} - q_{k-1}\sqrt{A})^m}{2\sqrt{A}} \end{cases}$$

manifesto aequationi $x^2 - Ay^2 = (-1)^{mk}$ satisfaciant, atque fractiones convergentes sunt versus $\sqrt{A}$, Ill. Legendrius nulli dubitationi obnoxium esse arbitrari videtur, valores eos, quos numeri x, y pro m = 2, 3, 4 etc. induant, ipsas esse fractiones convergentes in secunda, tertia, quarta periodo etc. positas. Quod quum argumentatione nostra minime explanetur, utilissimum esse visum est, dubitationem hujus rei eximere ac demonstrare fractionem $\frac{x}{y}$, cujus numerator et denominator sint valores (a), semper fractionem convergentem versus radicem $\sqrt{A}$ eam fore, quae paenultimo m^{tae} periodi quotienti respondeat.

Ad hunc finem problema nobis generale proponimus:

„per fractiones convergentes $\frac{p_\lambda}{q_\lambda}, \frac{p_{mk-1}}{q_{mk-1}}$, quarum altera in prima, altera „in m^{ta} periodo est, fractionem convergentem $\frac{p_{mk+\lambda}}{q_{mk+\lambda}}$, quae in $m+1^{ta}$ periodo, determinandi."

Quum sit $a_k = 2a$, ergo

$$a_k + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots + \cfrac{1}{a_\lambda}}} = \frac{p_\lambda + a q_\lambda}{q_\lambda}$$

erit
$$\frac{p_{mk+\lambda}}{q_{mk+\lambda}} = \frac{p_{mk-1}\left(\dfrac{p_\lambda + aq_\lambda}{q_\lambda}\right) + p_{mk-2}}{q_{mk-1}\left(\dfrac{p_\lambda + aq_\lambda}{q_\lambda}\right) + q_{mk-2}} = \frac{p_{mk-1}p_\lambda + (a p_{mk-1} + p_{mk-2})\, q_\lambda}{q_{mk-1}q_\lambda + (a q_{mk-1} + q_{mk-2})\, q_\lambda}$$

*) Théorie des nombres, pag 56. sqq.

Est autem $a\,p_{mk-1} + p_{mk-2} = A\,q_{mk-1}$, $a\,q_{mk-1} + q_{mk-2} = p_{mk-1}$, quod facile patebit, ideoque habetur

$$\frac{p_{mk+\lambda}}{q_{mk+\lambda}} = \frac{p_{mk-1}\,p_\lambda + q_{mk-1}\,A q_\lambda}{q_{mk-1}\,p_\lambda + q_{mk-1}\,q_\lambda}$$

Qua aequatione exhibita per $\frac{\alpha}{\beta} = \frac{\alpha'}{\beta'}$, ubi α, β numeri inter se primi, sequenti modo probari potest, esse $\alpha = \alpha'$, $\beta = \beta'$.

Facile reperitur $\alpha'^2 - \beta'^2 = (p_\lambda^2 - Aq_\lambda^2)\,(p_{mk-1}^2 - Aq_{mk-1}^2) = (-1)^{mk}\,(p_\lambda^2 - Aq_\lambda^2)$. Quodsi est $\alpha' = \delta\alpha$, $\beta' = \delta\beta$, habetur $\alpha'^2 - \beta'^2 = \delta^2\,(\alpha^2 - \beta^2)$, ergo $\delta^2\,(\alpha^2 - \beta^2) = (-1)^{mk}\,(p_\lambda^2 - Aq_\lambda^2)$. Atqui est $\alpha^2 - \beta^2 = (-1)^{mk}\,(p_\lambda^2 - Ap_\lambda^2)$, unde $\delta = 1$, i. e. $\alpha = \alpha'$, $\beta = \beta'$.

Itaque habemus

$$1.\quad p_{mk+\lambda} = p_{mk-1}\,p_\lambda + q_{mk-1}\,Aq_\lambda$$
$$2.\quad q_{mk+\lambda} = q_{mk-1}\,p_\lambda + p_{mk-1}\,q_\lambda,$$

quae aequationes pro $\lambda = {}_{k-1}$ in sequentes transeunt:

$$p_{(m+1)k-1} = p_{mk-1}\,p_{k-1} + q_{mk-1}\,Aq_{k-1}$$
$$q_{(m+1)k-1} = q_{mk-1}\,p_{k-1} + p_{mk-1}\,q_{k-1},$$

vel, si brevitatis gratia ponimus $p_{mk-1} = g_m$, $q_{mk-1} = h_m$, $p_{k-1} = g$, $q_{k-1} = h$:

$$1)\quad g_{m+1} = g\,g_m + h\,A\,h_m$$
$$2)\quad h_{m+1} = g\,h_m + h\,g_m.$$

Adjumento harum aequationum functionem inter fractiones $\frac{g}{h}$, $\frac{g_m}{h_m}$, quaeramus.

Methodus prima. Si ponimus succ. $m = 1$, $m = 2$, $m = 3$, etc., valores sequentes prodibunt

$g_2 = g^2 + Ah^2$	$h_2 = 2gh$
$g_3 = g^3 + 3gAh^2$	$h_3 = 3g^2h + Ah^3$
$g_4 = g^4 + 6g^2Ah^2 + A^2h^4$	$h_4 = 4g^3h + 4gAh^3$
$g_5 = g^5 + 10g^3Ah^2 + 5gA^2h^4$	$h_5 = 5g^4h + 10g^2Ah^3 + A^2h^5$
etc.	etc.

Unde facile perspicitur, numeros g_m, h_m has formas induere:

$$3.\quad g_m = g^m + K_{m,2}\,g^{m-2}Ah^2 + K_{m,4}\,g^{m-4}A^2h^4 + K_{m,6}\,g^{m-6}A^3h^6 + \text{etc.}$$
$$4.\quad h_m = L_{m,1}\,g^{m-1}h + L_{m,3}\,g^{m-3}Ah^3 + L_{m,5}\,g^{m-5}A^2h^5 + \text{etc.},$$

ubi $K_{m,2}$, $K_{m,4}$, etc., $L_{m,1}$, $L_{m,3}$, etc. sunt functiones quaedam numeri m, quas determinari oportet.

Quem ad finem mutationem videamus, quam functiones $K_{m,n}$, $L_{m,n}$ subeunt, quando m in $m + 1$ mutatur.

Quodsi primum in aequationibus 3, 4 pro m ponitur $m + 1$, ac deinde g_{m+1}, h_{m+1} aequationibus 1, 2, 3, 4, determinantur, hae relationes habebuntur

$K_{m+1,2} = K_{m,2} + L_{m,1}$	$L_{m+1,1} = L_{m,1} + 1$
$K_{m+1,4} = K_{m,4} + L_{m,3}$	$L_{m+1,3} = L_{m,3} + K_{m,2}$
$K_{m+1,6} = K_{m,6} + L_{m,5}$	$L_{m+1,5} = L_{m,5} + K_{m,4}$,
etc.	etc.

quae quum eodem modo ab se invicem dependeant ut coefficientes binomiales, videamus necesse est, utrum functiones nostrae revera coefficientes binomiales sint pro $m = 2$. Quod verum est, quia $g_2 = g^2 + Ah^2$, $h_2 = 2gh$, ideoque $K_{2,2} = 1 = 2_2$, $L_{2,1} = 2 = 2_1$.

Itaque habemus in genere $K_{m,2\delta} = m_{2\delta}$, $L_{m,2\delta+1} = m_{2\delta+1}$, unde ob aequationes 3, 4:

5. $g_m = g^m + m_2 g^{m-2} A h^2 + m_4 g^{m-4} A^2 h^4 + m_6 g^{m-6} A^3 h^6 + \text{etc.}$

6. $h_m = m_1 g^{m-1} h + m_3 g^{m-3} A h^3 + m_5 g^{m-5} A^2 h^5 + \text{etc.}$

Quibus valoribus, sub forma irrationali ita exhibitis

$$7.\quad g_m = \frac{(g + h\sqrt{A})^m + (g - h\sqrt{A})^m}{2}$$

$$8.\quad h_m = \frac{(g + h\sqrt{A})_m - (g - h\sqrt{A})^m}{2\sqrt{A},}$$

in aequationibus 1) et 2) substitutis, numeri $p_{mk+\lambda}$, $q_{mk+\lambda}$ habebuntur functiones numerorum p_{k-1}, q_{k-1}, p_λ, q_λ.

Methodus secunda. Exhibeantur aequationes $g_2 = g^2 + Ah^2$, $h_2 = 2gh$, sub forma irrationali

$$2g_2 = (g + h\sqrt{A})^2 + (g - h\sqrt{A})^2$$
$$2h_2\sqrt{A} = (g + h\sqrt{A})^2 - (g - h\sqrt{A})^2.$$

Multiplicando primam aequationem per g, secundam per $h\sqrt{A}$, ex relat. 1) prodibit $2g_3 = (g + h\sqrt{A})^3 + (g - h\sqrt{A})^3$; multiplicando vero secundam per g, primam per $h\sqrt{A}$ fit ex relat. 2) $2h_3\sqrt{A} = (g + h\sqrt{A})^3 - (g - h\sqrt{A})^3$.

Manifesto ex ultimis duabus aequationibus simili modo, ut ipsae ortae sunt, hae novae prodibunt $2g_4 = (g + h\sqrt{A})^4 + (g - h\sqrt{A})^4$; $2h_4\sqrt{A} = (g + h\sqrt{A})^4 - (g - h\sqrt{A})^4$ et sic porro.

Itaque revera in genere aequationes 7. et 8. habentur.

Sectio II.

De methodo radicis secundi gradus $\sqrt{A}$ in fractionem continuam evolvendae commodissima.

In opere Lambertiano, quod inscribitur „Beiträge zum Gebrauch der Mathematik," methodum olim legi, radicem $\sqrt{A}$, denotante A integrum quencunque, in fractionem continuam evolvendi, quam quidem utpote commodissimam accuratius in hac sectione perscrutaturus sum.

Discerpta etenim radice $\sqrt{A}$ in summam $a + y$, ubi intelligitur a maximus integer radice comprehensus, habetur $A = a^2 + 2ay + y^2$, unde $y = \frac{A - a^2}{2a + y}$, ideoque

$$1.\quad \sqrt{A} = a + \cfrac{A - a^2}{2a + \cfrac{A - a^2}{2a + \cfrac{A - a^2}{2a + \dots\dots + \cfrac{A - a^2}{2a + y,}}}}$$

ubi multitudo terminorum fractionis continuae indeterminata est.

1.

Primum disquirendum est, num haec fractio continua, residuo y neglecto, in infinitum continuata ad limitem $\sqrt{A}$ convergat. Consideremus ideo fractionem continuam generaliorem

$$1)\quad x = a + \cfrac{\alpha_1}{a_1 + \cfrac{\alpha_2}{a_2 + \ldots + \cfrac{\alpha_n}{a_n + y_n,}}}$$

et designemus fractiones partiarias per $\frac{p}{q}, \frac{p_1}{q_1}, \frac{p_2}{q_2}$, etc.

Notum est esse $x = \frac{p_{n-1}(a_n + y_n) + p_{n-2}\alpha_n}{q_{n-1}(a_n + y_n) + q_{n-2}\alpha_n}$, ergo, quum sit $p_{n-1}a_n + p_{n-2}\alpha_n = p_n$, $q_{n-1}a_n + q_{n-2}\alpha_n = q_n$:

$$2.\quad x = \frac{p_n + p_{n-1}y_n}{q_n + q_{n-1}y_n}$$

Unde manat

$$3.\quad \delta_n = x - \frac{p_n}{q_n} = \frac{(p_{n-1}q_n - p_n q_{n-1})y_n}{q_n(q_n + q_{n-1}y_n)}$$

$$4.\quad \delta_{n-1} = x - \frac{p_{n-1}}{q_{n-1}} = \frac{p_n q_{n-1} - p_{n-1}q_n}{q_{n-1}(q_n + q_{n-1}y_n)},$$

ideoque

$$5.\quad \frac{\delta_n}{\delta_{n-1}} = -\frac{q_{n-1}}{q_n}y_n.$$

Hinc facile sequitur

$$\delta_n = (-1)^{n-1}\delta_1 \cdot \frac{q_1}{q_n} \cdot y_2 y_3 y_4 \ldots y_n.$$

Atqui est $\delta_1 = x - \frac{p_1}{q_1} = a + \frac{\alpha_1}{a_1 + y_1} - \frac{a a_1 + \alpha_1}{a_1} = -\frac{\alpha_1 + y_1}{a_1(a_1 + y_1)}$, $q_1 \delta_1 = -\frac{\alpha_1 y_1}{a_1 + y_1}$; porro $y = \frac{\alpha_1}{a_1 + y_1}$, $y y_1 = \frac{\alpha_1 y_1}{a_1 + y_1}$, unde $y y_1 = -q_1\delta_1$, ergo ex praecedentibus

$$6.\quad \delta_n = (-1)^n \cdot \frac{1}{q_n} \cdot y y_1 y_2 \ldots y_n$$

Quodsi ponimus $x = \sqrt{A}$ atque $y = y_1 = y_2 = \text{etc.} = \sqrt{A} - a$, ex formula 6. erit

$$7.\quad \sqrt{A} - \frac{p_n}{q_n} = (-1)^n \cdot \frac{1}{q_n}(\sqrt{A} - a)^{n+1}$$

Quum jam sit y fractio genuina atque manifesto q_n in infinitum crescat, differentia $\sqrt{A} - \frac{p_n}{q_n}$, indice n in infinitum tendente, non solum semper diminuetur, sed cifram limitem habebit, unde

$$8.\quad \sqrt{A} = a + \cfrac{A-a^2}{2a + \cfrac{A-a^2}{2a + \cfrac{A-a^2}{2a + \text{ in infinitum.}}}}$$

Ceterum quia est $\sqrt{A} - a < \frac{A-a^2}{2a}$, erit respectu valoris absoluti

$$\sqrt{A} - \frac{p_n}{q_n} < \frac{1}{q_n} \cdot \left(\frac{A-a^2}{2a}\right)^{n+1}$$

2.

Valores independentes fractionum convergentium hoc modo inveniuntur: Quando in relatione 2. accipitur $y_n = \sqrt{A} - a$, $x = \sqrt{A}$, habetur

$$(q_n - a q_{n-1} - p_{n-1}) \sqrt{A} = p_n - a p_{n-1} - q_{n-1} A,$$

unde

$$9.\quad p_n = a p_{n-1} + A q_{n-1}.$$
$$10.\quad q_n = a q_{n-1} + p_{n-1}.$$

Quum hae aequationes ex aequationibus 1) et 2) Sect. I. prodeant ponendo $g = a$, $h = 1$, $m = n - 1$, propter relationes 7., 8. Sect. I. valores independentes habentur

$$11.\quad p_{n-1} = \frac{(a+\sqrt{A})^n + (a-\sqrt{A})^n}{2}$$

$$12.\quad q_{n-1} = \frac{(a+\sqrt{A})^n - (a-\sqrt{A})^n}{2\sqrt{A}}$$

Ex aequationibus 9., 10. manat

$$13.\quad p_n q_{n-1} - p_{n-1} q_n = -(p_{n-1}^2 - A q_{n-1}^2)$$

Porro est, ut facile invenitur,

$$p_n = 2a \,.\, p_{n-1} + (A-a^2) p_{n-2}$$
$$q_n = 2a \,.\, q_{n-1} + (A-a^2) q_{n-2},$$

unde $p_n q_{n-1} - p_{n-1} q_n = -(A-a^2)(p_{n-1} q_{n-2} - p_{n-2} q_{n-1})$. Quum jam sit $p_1 q - p q_1 = A - a^2$, erit

$$14.\quad p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}(A-a^2)^n,$$

ergo ex relat. 13.

$$15.\quad p_{n-1}^2 - A q_{n-1}^2 = (-1)^n (A-a^2)^n = (a^2 - A)^n.$$

3.

Quando $A - a^2 = 1$, fractio continua 1. in fractionem continuam simplicem

$$\sqrt{A} = a + \cfrac{1}{2a + \cfrac{1}{2a + \text{ in inf.}}}$$

transibit.

Vice versa si fractionis continuae simplicis periodus unicum terminum comprehendit, semper aequatio locum habebit $A - a^2 = 1$.

Est enim $\sqrt{A} = a + \frac{1}{2a + (\sqrt{A} - a)} = a + \frac{1}{a + \sqrt{A}}$, unde $(\sqrt{A} - a)(\sqrt{A} + a) = 1$, i. e. $A - a^2 = 1$.

Praebebunt igitur radices

$$\sqrt{2},\ \sqrt{5},\ \sqrt{10},\ \sqrt{17},\ \sqrt{26},\ \sqrt{37},\ \sqrt{50},\ \sqrt{65},\ \text{etc.}$$

fractiones continuas simplices, quarum periodus unum modo terminum habet.

4.

Quoties in fractione continua 1. numerator $A - a^2$ et denominator $2a$ divisorem comm. max. habent, in aliam transformari potest ea indole praeditam, ut numerator et denominator termini cujusvis numeri inter se primi sint. Methodus huc spectans sequenti propositione nititur.

„Fractionis alicujus continuae fractiones convergentes eosdem valores „retinent, quando numerator et denominator termini cujusvis nec non „numerator proxime sequentis termini per eandem quantitatem multipli- „catur vel dividitur."

Exempli gratia est

$$a + \cfrac{\alpha_1}{a_1 + \cfrac{\alpha_2}{a_2 + \cfrac{\alpha_3}{a_3 + \cfrac{\alpha_4}{a_4 + \cfrac{\alpha_5}{a_5 + \text{etc.}}}}}} = a + \cfrac{\alpha_1}{a_1 + \cfrac{\alpha_2}{a_2 + \cfrac{z\alpha_3}{z a_3 + \cfrac{z\alpha_4}{a_4 + \cfrac{\alpha_5}{a_5 + \text{etc.}}}}}}$$

Quod ita demonstro.

Denotemus fractiones convergentes ambarum fractionum continuarum resp. per

$$\frac{p}{q},\ \frac{p_1}{q_1},\ \frac{p_2}{q_2},\ \text{etc.}\qquad \frac{P}{Q},\ \frac{P_1}{Q_1},\ \frac{P_2}{Q_2},\ \text{etc.}$$

Primum per se patet, esse $\frac{P}{Q} = \frac{p}{q}$, $\frac{P_1}{Q_1} = \frac{p_1}{q_1}$, $\frac{P_2}{Q_2} = \frac{p_2}{q_2}$, ita ut in aequatione quaque et numeratores et denominatores sint aequales.

Porro est

$$p_3 = p_2 a_3 + p_1 \alpha_3 \qquad\qquad q_3 = q_2 a_3 + q_1 \alpha_3$$

$$P_3 = P_2(z a_3) + P_1(z\alpha_3) = (p_2 a_3 + p_1 \alpha_3) z = p_3 z \qquad\Big|\qquad Q_3 = Q_2(z a_3) + Q_1(z a_3) = (q_2 a_3 + q_1 \alpha_3) z = q_3 z,$$

unde

$$\frac{P_3}{Q_3} = \frac{p_3 z}{q_3 z} = \frac{p_3}{q_3}.$$

Deinde est

$P_4 = P_3 a_4 + P_2 (z\alpha_4) = (p_3 a_4 + p_2 \alpha_4) z = p_4 z$ | $Q_4 = Q_3 a_4 + Q_2 (z\alpha_4) = (q_3 a_4 + q_2 \alpha_4) z = q_4 z,$

unde

$$\frac{P_4}{Q_4} = \frac{p_4 z}{q_4 z} = \frac{p_4}{q_4}.$$

Postremo est

$P_5 = P_4 a_5 + P_3 \alpha_5 = (p_4 a_5 + p_3 \alpha_5) z = p_5 z$ | $Q_5 = Q_4 a_5 + Q_3 \alpha_5 = (q_4 a_5 + q_3 \alpha_5) z = q_5 z$

unde

$$\frac{P_5}{Q_5} = \frac{p_5 z}{q_5 z} = \frac{p_5}{q_5}.$$

Eodem modo perspicitur fore

$$\frac{P_6}{Q_6} = \frac{p_6 z}{q_6 z} = \frac{p_6}{q_6}, \qquad \frac{P_7}{Q_7} = \frac{p_7 z}{q_7 z} = \frac{p_7}{q_7}, \text{ etc.}$$

5.

Quodsi in fractione continua 1. $A - a^2$ metitur $2a$, vel si est

$$\frac{2a}{A - a^2} = b$$

ex theoremate praecedente erit

$$\sqrt{A} = a + \cfrac{1}{b + \cfrac{1}{2a + \cfrac{1}{b + \cfrac{1}{2a + \text{ in inf.,}}}}}$$

unde fractio continua simplex orta est, cujus periodus duos terminos habet.

Vice versa si fractio continua simplex periodica ea indole praedita est, ut periodus duos terminos habeat, semper aequatio locum habebit $\frac{2a}{A - a^2} = b$.

Est enim $p_1^2 - A q_1^2 = 1$, atqui $p_1 = ab + 1$, $q_1 = b$, ergo $(ab + 1)^2 - Ab^2 = 1$, $(a^2 - A) b^2 + 2ab = o$, $(A - a^2) b - 2a = o$.

Ex. gr. pro $A = 12$ est $a = 3$, $A - a^2 = 3$, $2a = 6$, $b = 2$, ideoque

$$\sqrt{12} = 3 + \cfrac{1}{2 + \cfrac{1}{6 + \cfrac{1}{2 + \cfrac{1}{6 \text{ cett.}}}}}$$

Ceterum numerus A ejusmodi, ut radix $\sqrt{A}$ fractionem praebeat continuam, cujus periodus duos terminos habeat, facile inveniri potest. Est enim

$$A = a^2 + \frac{2a}{b},$$

quo loco numeri a, b restrictioni obnoxii sunt, ut b ipsum 2a metiatur.

Ex. gr. pro $b = 3$ numerus a valores induere potest 3, 6, 9, 12, etc.; hinc $\frac{2a}{b} = 2, 4, 6, 8$, etc., ergo

$$A = 11.\ 40.\ 87.\ 152.\ 235.\ \text{etc.}$$

6.

Superest, ut casum disquiramus, in quo numeri $A - a^2$, 2a divis. comm. max. ab ipso $A - a^2$ diversum admittant. Quod si eveniat, fractio continua Lambertiana methodo exorta cum fractione continua simplici respondente minime congruit. Sed tamen formam simpliciorem accipiet, si theorema §. 4. ad eam applicatur.

Divisor comm. max. numerorum $A - a^2$, 2a sit ϑ atque

$$A - a^2 = \beta\vartheta,\quad 2a = b\vartheta;$$

tum erit ex §. 4.

$$\sqrt{A} = a + \cfrac{\beta}{b + \cfrac{\beta}{2a + \cfrac{\beta}{b + \cfrac{\beta}{2a + \text{etc.}}}}}$$

Quodsi β et 2a habeant divisorem comm. max. ϑ_1, ita ut sit

$$\beta = \gamma\vartheta_1,\quad 2a = c\vartheta_1,$$

erit

$$\sqrt{A} = a + \cfrac{\beta}{b + \cfrac{\gamma}{c + \cfrac{\gamma}{b + \cfrac{\gamma}{c + \text{cett.}}}}}$$

Si porro γ et b habeant divisorem comm. max. ϑ_2, ita ut sit

$$\gamma = \delta\vartheta_2,\quad b = d\vartheta_2,$$

erit

$$\sqrt{A} = a + \cfrac{\beta}{b + \cfrac{\gamma}{c + \cfrac{\delta}{d + \cfrac{\delta}{c + \cfrac{\delta}{d + \text{cett.}}}}}}$$

Si deinde ϑ_3 divisor comm. maximus numerorum δ et c, vel

$$\delta = \varepsilon\vartheta_3,\quad c = e\vartheta_3,$$

habetur

$$\sqrt{A} = a + \cfrac{\beta}{b + \cfrac{\gamma}{c + \cfrac{\delta}{d + \cfrac{\varepsilon}{e + \cfrac{\varepsilon}{d + \cfrac{\varepsilon}{e + \text{cett.},}}}}}}$$

ubi jam β ad b, γ ad c, δ ad d, ε ad e primus est.

Quum autem sit $\beta > \gamma > \delta > \varepsilon$, etc., i. e. numeratores continuo decrescant, divisor comm. max. tandem unitas fiet, ex quo methodi nostrae applicatione tandem fractio continua evadet, in qua numerator et denominator termini cujusvis sint numeri inter se primi.

Ex. gr. $A = 44$, $a = 6$, $A - a^2 = 8$, $2a = 12$, ergo $\delta = 4$, $\beta = 2$, $b = 3$, unde $\vartheta_1 = 2$, $\gamma = 1$, $c = 6$, $\vartheta_2 = 1$, ergo

$$\sqrt{44} = 6 + \cfrac{2}{3 + \cfrac{1}{6 + \cfrac{1}{3 + \cfrac{1}{6 + \cfrac{1}{3 + \text{etc.}}}}}}$$

Sectio III.

Connexus fractionum convergentium, quae in fractione continua simplici, evolutione radicis aequationis secundi gradus exorta, eidem periodorum quotienti completo respondeant.

Quomodo radix aequationis secundi gradus

$$f x^2 + g x + h = o$$

scilicet

$$x = \frac{-g + \sqrt{(g^2 - 4fh)}}{2f}$$

in fractionem continuam simplicem periodicam evolvi possit, Ill. Legendrius in egregio opere, quod inscribitur Théorie des nombres, Tom. I., p. 81. sqq. edocuit.

Maximi momenti est, expressionem generalem fractionum convergentium reperire, quae eidem periodorum quotienti completo respondeant. Quod attinet ad hujus problematis solutionem, animadversiones nonnullas adjicere liceat, quae ut intelligantur, de omnibus, quibus haec superstruxi, Legendrii opus evolvendum est.

Si periodum quotientium, quae ex evolutione quotientis completi $\frac{\sqrt{A} + J}{D}$ prodeat, designamus per

$$\mu, \mu', \mu'', \mu''', \dots \omega.$$

fractionem convergentem quotienti μ proxime antecedentem per $\frac{p}{q}$, fractiones convergentes ultimas primae, secundae, tertiae periodi etc. per $\frac{p^{(1)}}{q^{(1)}}, \frac{p^{(2)}}{q^{(2)}}, \frac{p^{(3)}}{q^{(3)}}$, etc. fractionem convergentem ipsi $\frac{p}{q}$ proxime antecedentem per $\frac{p_0}{q_0}$, fractionem denique continuam finitam

$$\mu + \cfrac{1}{\mu' + \cfrac{1}{\mu'' + \dots + \cfrac{1}{\omega}}}$$

per $\frac{\alpha}{\beta}$, habebitur

$$1.\quad \frac{p^{(1)}}{q^{(1)}} = \frac{p\left(\frac{\alpha}{\beta}\right) + p_0}{q\left(\frac{\alpha}{\beta}\right) + q_0} = \frac{p\alpha + p_0\beta}{q\alpha + q_0\beta}.$$

Est vero $x = \frac{\sqrt{A} - \frac{1}{2}g}{f} = \frac{p\left(\frac{\sqrt{A}+J}{D}\right) + p_0}{q\left(\frac{\sqrt{A}+J}{D}\right) + q_0} = \frac{p(\sqrt{A}+J) + p_0 D}{q(\sqrt{A}+J) + q_0 D}$,

unde
$$qJ + q_0 D - \tfrac{1}{2}gq - fp = 0$$
$$fpJ + fp_0 D - qA + \tfrac{1}{2}g\,(qJ + q_0 D) = 0,$$

ideoque, quum sit (Théorie des nombres) $A = \frac{1}{4}g^2 - fh$:

$$2.\quad \begin{cases} p_0 = -\dfrac{p}{D}\,(\tfrac{1}{2}g + J) - \dfrac{hq}{D} \\[2ex] q_0 = +\dfrac{q}{D}\,(\tfrac{1}{2}g - J) + \dfrac{fp}{D} \end{cases}$$

Quibus valoribus substitutis in relat. 1. prodit, si brevitatis gratia ponimus

$$3.\quad \alpha - \frac{\beta J}{D} = \varphi,\ \frac{\beta}{D} = \psi:$$

$$4.\quad \frac{p^{(1)}}{q^{(1)}} = \frac{p(\varphi - \frac{1}{2}g\psi) - qh\psi}{q(\varphi + \frac{1}{2}g\psi) + pf\psi}.$$

Hinc Legendrius viam sequentem ingressus est: Propter periodorum identitatem est

$$5.\quad \frac{p^{(2)}}{q^{(2)}} = \frac{p^{(1)}(\varphi - \frac{1}{2}g\psi) - q^{(1)}h\psi}{q^{(1)}(\varphi + \frac{1}{2}g\psi) + p^{(1)}f\psi}$$

Ex aequationibus 4. et 5. facile prodeunt hac relationes

$$6.\quad \begin{cases} p^{(2)} = 2\varphi p^{(1)} - \varepsilon p \\ q^{(2)} = 2\varphi q^{(1)} - \varepsilon q, \end{cases}$$

ubi brevitatis gratia $\varepsilon = \varphi^2 - A\psi^2$.

Haec lex est, ex qua tres fractiones convergentes, quae se proxime sequuntur, ab se invicem dependeant.

Numeratores $p, p^{(1)}, p^{(2)}, \ldots$ nec non denominatores $q, q^{(1)}, q^{(2)}, \ldots$ seriem recurrentem constituunt, cujus scala relationis est $2\varphi, -\varepsilon$. Quarum serierum doctrina innixus Legendrius statim expressionem generalem fractionum convergentium deduxit.*)

Equidem finem mihi proposui, ut ad illarum serierum doctrinam non provocans methodum edoceam, ex qua problema nostrum facillime solvatur. Et initium quidem ab aequatione 4. capiendum, ex qua in genere

$$7.\quad \begin{cases} p^{(n+1)} = p^{(n)}(\varphi - \tfrac{1}{2}g\varphi) - q^{(n)}h\psi \\ q^{(n+1)} = q^{(n)}(\varphi + \tfrac{1}{2}g\psi) + p^{(n)}f\psi \end{cases}$$

Manifesto $p^{(n)}, q^{(n)}, p^{(n+1)}, q^{(n+1)}$ formas induent

$$8.\quad \begin{cases} p^{(n)} = K_n p - L_n q, \\ q^{(n)} = N_n q + M_n p, \end{cases} \qquad 9.\quad \begin{cases} p^{(n+1)} = K_{n+1} p - L_{n+1} q \\ q^{(n+1)} = N_{n+1} q - M_{n+1} p \end{cases}$$

Substitutis valoribus $p^{(n)}, q^{(n)}$ ex rel. 8. in rel. 7., prodibit

$$p^{(n+1)} = p\{K_n(\varphi - \tfrac{1}{2}g\psi) - M_n\psi h\} - q\{L_n(\varphi - \tfrac{1}{2}g\psi) - N_n\psi h\},$$
$$q^{(n+1)} = q\{N_n(\varphi - \tfrac{1}{2}g\psi) - L_n\psi f\} + p\{M_n(\varphi - \tfrac{1}{2}g\psi) + K_n\psi f\}.$$

Quae relationes si cum relat. 9 conferantur, habebuntur hae aequationes

$$10.\quad \begin{cases} K_{n+1} = K_n(\varphi - \tfrac{1}{2}g\psi) - M_n\psi h, \\ L_{n+1} = L_n(\varphi - \tfrac{1}{2}g\psi) + N_n\psi h, \end{cases} \qquad 11.\quad \begin{cases} N_{n+1} = N_n(\varphi + \tfrac{1}{2}g\psi) - L_n\psi f. \\ M_{n+1} = M_n(\varphi + \tfrac{1}{2}g\psi) - K_n\psi f. \end{cases}$$

Jam ut coefficientes K, L, M, N determinemus, a valoribus $K_1 = \varphi - \tfrac{1}{2}g\psi$, $L_1 = \psi h$, $N_1 = \varphi + \tfrac{1}{2}g\psi$, $M_1 = \psi f$, qui manant ex relatt. 7, initium capiendum est; hinc succ. adjumento relatt. 10. 11. ad valores K_2, L_2, M_2, N_2; K_3, L_3, M_3, N_3, etc., ascendi potest.

Animadvertens igitur relationem $A = \tfrac{1}{4}g^2 - fh$ aequationes sequentes habebis:

$$K_2 = (\varphi - \tfrac{1}{2}g\psi)(\varphi - \tfrac{1}{2}g\psi) - fh\psi\psi = \varphi\varphi + A\psi\psi - \tfrac{1}{2}g(\varphi\psi + \psi\varphi) = \varphi_2 - \tfrac{1}{2}g\psi_2;$$
$$L_2 = h\psi(\varphi - \tfrac{1}{2}g\psi) + h\psi(\varphi + \tfrac{1}{2}g\psi) = h(\varphi\psi + \psi\varphi) = h\psi_2,$$
$$N_2 = (\varphi + \tfrac{1}{2}g\psi)(\varphi + \tfrac{1}{2}g\psi) - fh\psi\psi = \varphi\varphi + A\psi\psi + \tfrac{1}{2}g(\varphi\psi + \psi\varphi) = \varphi_2 + \tfrac{1}{2}g\psi_2,$$
$$M_2 = f\psi(\varphi + \tfrac{1}{2}g\psi) + f\psi(\varphi - \tfrac{1}{2}g\psi) = f(\varphi\psi + \psi\varphi) = f\psi_2,$$

ubi brevitatis gratia $\varphi_2 = \varphi\varphi + A\psi\psi$, $\psi_2 = \varphi\psi + \psi\varphi$.

Deinde erit

$$K_3 = (\varphi_2 - \tfrac{1}{2}g\psi_2)(\varphi - \tfrac{1}{2}g\psi) - fh\psi_2\psi = \varphi\varphi_2 + A\psi\psi_2 - \tfrac{1}{2}g(\varphi\psi_2 + \psi_2\varphi) = \varphi_3 + \tfrac{1}{2}g\psi_3,$$
$$L_3 = h\psi_2(\varphi - \tfrac{1}{2}g\psi) + h\psi(\varphi_2 + \tfrac{1}{2}g\psi_2) = h(\varphi\psi_2 + \psi\varphi_2) = h\psi_3,$$
$$N_3 = (\varphi_2 + \tfrac{1}{2}g\psi_2)(\varphi + \tfrac{1}{2}g\psi) - fh\psi_2\psi = \varphi\varphi_2 + A\psi\psi_2 + \tfrac{1}{2}g(\varphi\psi_2 + \psi_2\varphi) = \varphi_3 + \tfrac{1}{2}g\psi_3,$$
$$M_3 = f\psi_2(\varphi + \tfrac{1}{2}g\psi) + f\psi(\varphi_2 - \tfrac{1}{2}g\psi_2) = f(\varphi\psi_2 + \psi\varphi_2) = f\psi_3,$$

ubi brevitatis gratia $\varphi\varphi_2 + A\psi\psi_2 = \varphi_3$, $\varphi\psi_2 + \varphi_2\psi = \psi_3$.

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*) Théorie des nombres, pag. 56.

„Or il résulte de la théorie connue de ces suites, que si l'on fait $(\varphi + \psi\sqrt{A})^n = \Phi + \Psi\sqrt{A}$, „n étant un entier quelconque, le terme général demandé $\frac{p^{(n)}}{q^{(n)}}$, sera donné par les formules

$$p^{(n)} = a'\Phi + b'\Psi$$
$$q^{(n)} = a''\Phi + b''\Psi$$

„où il ne reste plus à déterminer que les coefficiens a', b', a'', b'', etc."

Quam quidem ratiocinationem si accuratius perspexeris, omnino has relationes locum habere intelliges: $K_n = \varphi_n - \frac{1}{2}g\,\psi_n$, $L_n = h\psi_n$, $N_n = \varphi_n + \frac{1}{2}g\psi_n$, $M_n = f\psi_n$, ubi quantitates per φ, ψ designatae ita ab se invicem dependeant, ut sit

$$12.\quad \begin{cases} \varphi_{n+1} = \varphi\varphi_n + A\psi\psi_n \\ \psi_{n+1} = \varphi\psi_n + \psi\varphi_n \end{cases}$$

Itaque relatt. 8. in sequentes mutantur

$$13.\quad \begin{cases} p^{(n)} = p(\varphi_n - \tfrac{1}{2}g\psi_n) - hq\psi_n \\ q^{(n)} = p(\varphi_n + \tfrac{1}{2}g\psi_n) + fp\psi_n, \end{cases}$$

quae quidem cum Legendrianis l. c. pag. 87. congruunt.

Ceterum problema, quantitates φ_n, ψ_n adjumento relatt. 12. functiones ipsarum φ, ψ exprimendi in sectione I. jam solvimus, unde est

$$14.\quad \begin{cases} \varphi_n = \dfrac{(\varphi + \psi\sqrt{A})^n + (\varphi - \psi\sqrt{A})^n}{2} \\ \psi_n = \dfrac{(\varphi + \psi\sqrt{A})^n - (\varphi - \psi\sqrt{A})^n}{2\sqrt{A}} \end{cases}$$

Quoniam numeri $p^{(n)}$, $q^{(n)}$ non aliunde pendentes expressi sunt, nisi de numeris p, q, propositum plane consecutus sum.

Sectio IV.

Disquisitiones nonnullae ad aequationem $p^2 - Aq^2 = 1$ spectantes.

Quomodo aequationis $p^2 - Aq^2 = 1$ resolutio cum evolutione radicis secundi gradus $\sqrt{A}$ cohaerescat, jam in sectione I. in memoriam revocavi. Hanc aequationem, ita spectatam, ut p, q sint minimi ei satisfacientes numeri, i. e. fractio $\frac{p}{q}$ ad primam vel ad secundam periodum pertineat, prout multitudo periodi terminorum par vel impar, principiis mere arithmeticis perscrutans, non modo numerorum p, q insignes proprietates cognovi, verum etiam, unico tantum casu excepto, criterium quoddam reperi, utrum multitudo periodi terminorum par an impar sit.

Ill. Legendrium *) similiter quidem aequationem nostram perscrutatum esse, peritum non effugiet, sed ex fonte longe altiore totam hanc rem consideravi multaque nova memoratu dignissima inde deducta esse, ex ipsis, quas traditurus sum, disquisitionibus meis elucebit.

I. Inquiratur in aequationem $p^2 - Aq^2 = 1$, ubi A intelligitur num. quicunque impar.

1.

Caput rei resolutione aequationis in factores nititur, ita ut sub formam redigatur $(p+1)(p-1) = Aq^2$, quo facto disquisitio in duas partes distribuenda.

*) Théorie des nombres. Tom. I. §. VII.

(A.) Quoties p impar est, q vero par, habemus $\frac{1}{2}(p+1)\cdot\frac{1}{2}(p-1) = A.(\frac{1}{2}q)^2$. Quodsi est ϑ divisor comm. max. numerorum $\frac{1}{2}(p+1)$, $\frac{1}{2}q$, aequationis dextera pars per ϑ^2 divisibilis est, unde etiam sinistra, cujus factores differentiam 1 constituentes quum factorem ϑ non simul habeant, factor $\frac{1}{2}(p+1)$ per ϑ^2 divisibilis erit. Quia igitur $\frac{1}{2}(p+1)$, $\frac{1}{2}q$ formam resp. induunt $\vartheta^2\varrho_1$, $\vartheta q'$, habetur aequatio $\varrho_1\cdot\frac{1}{2}(p-1) = Aq'^2$. Atqui q' ad ϱ_1 primus est, unde q'^2 metitur $\frac{1}{2}(p-1)$, vel est $\frac{1}{2}(p-1) = q'^2\varrho_2$. Hinc aequatio prodit $\varrho_1\varrho_2 = A$.

Quo loco facile perspicietur, num. q' esse divisorem comm. max. numerorum $\frac{1}{2}(p-1)$, $\frac{1}{2}q$.

(B.) Quoties autem p est par, q vero impar, divisore comm. maximo numerorum $p+1$, q designato per ϑ, ita ut sit $p+1 = \vartheta^2\varrho_1$, $q = \vartheta q'$, erit $\varrho_1(p-1) = Aq'^2$, unde, quum q' ad ϱ_1 primus sit, $p-1 = q'^2\varrho_2$, ideoque $\varrho_1\varrho_2 = A$.

Ceterum perspicuum erit, ipsum q' divisorem communem maximum esse numerorum $p-1$, q.

Quodsi κατ᾽ ἐξοχήν divisor comm. max. numerorum $p+1$, q designatur per ϑ_1, divisor comm. max. numerorum $p-1$, q per ϑ_2, nascentur haec duo relationum systemata, quorum in altero p impar, q par, in altero p par, q impar est:

Systema primum.	Systema secundum.
$\frac{1}{2}(p+1) = (\frac{1}{2}\vartheta_1)^2\varrho_1$	$p+1 = \vartheta_1^2\sigma_1$
$\frac{1}{2}(p-1) = (\frac{1}{2}\vartheta_2)^2\varrho_2$	$p-1 = \vartheta_2^2\sigma_2$
$\frac{1}{2}\vartheta_1.\frac{1}{2}\vartheta_2 = \frac{1}{2}q$	$\vartheta_1\vartheta_2 = q$
$\varrho_1\varrho_2 = A$	$\sigma_1\sigma_2 = A$
$(\frac{1}{2}\vartheta_1)^2\varrho_1 = (\frac{1}{2}\vartheta_2)^2\varrho_2 = 1$	$\vartheta_1^2\sigma_1 = \vartheta_2^2\sigma_2 = 2$

Quas relationes si penitus perspexeris, sequentia facillime cognosces:

In systemate primo numeri ϱ_1, ϱ_2 ambo impares inter seque primi, ergo numerorum $\frac{1}{2}\vartheta_1$, $\frac{1}{2}\vartheta_2$ alter par, alter impar amboque inter se primi.

In systemate secundo numeri σ_1, σ_2 ambo impares inter seque primi, ergo ϑ_1, ϑ_2 ambo impares, et primi inter se.

In primo casu numerus ϱ nunquam unitas esse potest, quoniam, si hoc eveniret, haberetur $(\frac{1}{2}\vartheta_1)^2 - (\frac{1}{2}\vartheta_2)^2 A = 1$, ideoque p, q non essent minimi numeri aequationi $x^2 - Ay^2 = 1$ satisfacientes.

Postremo si est $\varrho_2 = 1$, vel aequatio locum habet $(\frac{1}{2}\vartheta_2)^2 - (\frac{1}{2}\vartheta_1)^2 A = -1$, multitudo periodi terminorum necessario impar erit.

2.

Utrum primum systema an secundum incidat, quod scire utilissimum est, ab indole numeri A atque ex parte a forma pendebit, quam sinistra pars aequationum $(\frac{1}{2}\vartheta_1)^2\varrho_1 - (\frac{1}{2}\vartheta_2)^2\varrho_2 = 1$, $\vartheta_1^2\sigma_1 - \vartheta_2^2\sigma_2 = 2$ in utroque casu induat. Quam formam facile reperies in memoriam revocans, quadratum numeri paris 2k formae esse 4k, imparis vero $4k \pm 1$ formae $8k+1$.

(A.) Hinc si $\frac{1}{2}\vartheta_1$ impar est, $\frac{1}{2}\vartheta_2$ par, ϱ_1 formam induet $4k+1$. Sin $\frac{1}{2}\vartheta_1$ par, $\frac{1}{2}\vartheta_2$ impar, ϱ_2 formam $4k+3$ induet.

Quoties igitur numerus A formae est $4m+1$, numeri ϱ_1, ϱ_2 ambo formae esse debent $4k+1$, quando $\frac{1}{2}\vartheta_1$ impar, $\frac{1}{2}\vartheta_2$ par, ambo vero formae $4k+3$, quando $\frac{1}{2}\vartheta_1$ par, $\frac{1}{2}\vartheta_2$ impar est.

Quoties vero numerus et formae est $4m+3$, numerus ϱ_1 formam induit $4k+1$, numerus ϱ_2 formam $4k+3$.

(B.) Quod attinet ad systema secundum, si numeri σ_1, σ_2 ambo formam $4k+1$, vel ambo formam $4k+3$ habent, differentia $\vartheta_1^2\sigma_1 - \vartheta_2^2\sigma_2$ formam induet $4k$, q. f. n.

Ex quo systema secundum locum habere nequit, nisi σ_1, σ_2 diversas formas habent, i. e. numerus A formae est $4m+3$.

Reperies praesertim, combinationes modo sequentes incidere posse, quoties aequatio habeatur $\vartheta_1^2\sigma_1 - \vartheta_2^2\sigma_2 = 2$

$\sigma_1 = 8k+1$	$\sigma_1 = 8k+5$	$\sigma_1 = 8k+3$	$\sigma_1 = 8k+7$
$\sigma_2 = 8k+7$	$\sigma_2 = 8k+3$	$\sigma_2 = 8k+1$	$\sigma_2 = 8k+5$
$A = 8m+7$	$A = 8m+7$	$A = 8m+3$	$A = 8m+3$

3.

Ex paragrapho antecedente haec theoremata principalia manant:

I. Quoties A formae est $4m+1$, primum tantum systema locum habet, atque ϱ_1, ϱ_2 eandem formam $4k+1$, vel $4k+3$ induunt, primam, si $\frac{1}{2}\vartheta_1$ impar, $\frac{1}{2}\vartheta_2$ par, secundam vero, si $\frac{1}{2}\vartheta_1$ par, $\frac{1}{2}\vartheta_2$ vero impar.

II. Quoties autem A formae est $4m+3$, tum primum, tum secundum systema locum habere potest, et si illud eveniat, ϱ_1 formae est $4k+1$, ϱ_2 formae $4k+3$, si hoc vero, combinationum, quas in 2. posuimus, aliqua exstabit.

4.

Disquisitio peculiaris formae $4m+1 = A$.

a) Si A potestas (impar) numeri primi $4m+1$ est, ob aequationem $\varrho_1\varrho_2 = A$, quum ϱ_1, ϱ_2 sint primi inter se, atque ϱ_1 unitas esse nequeat, necessario relationes habentur $\varrho_1 = A$, $\varrho_2 = 1$, unde manat aequatio

$$(\tfrac{1}{2}\vartheta_2)^2 - (\tfrac{1}{2}\vartheta_1)^2 A = -1,$$

ex qua

α) -1 residuum quadraticum potestatis numeri primi $4m+1$,

β) multitudo periodi terminorum in fractione continua ipsius $\sqrt{A}$ *impar*.

γ) Quotiente aliquo completo fract. cont. designato per $\frac{\sqrt{A}+J_n}{D_n}$, multitudine terminorum per $_k$, notum est esse $D_{\frac{1}{2}(k-1)} = D_{\frac{1}{2}(k+1)}$, unde ob aequationem $D_{\frac{1}{2}(k-1)}\, D_{\frac{1}{2}(k+1)} = A - J^2_{\frac{1}{2}(k+1)}$:

$$A = D^2_{\frac{1}{2}(k+1)} + J^2_{\frac{1}{2}(k+1)},$$

i. e. potestas impar numeri primi formae $4m+1$ in duo semper quadrata discerpi potest.

Hoc theorema pro casu, in quo A numerus primus est formae $4m+1$, similiter Ill. Legendrius *) probavit. Idem pro quacunque potestate impari numeri primi $4m+1$ valere, hunc geometram effugisse videtur.

b) Si numerus A factorem primum formae $4m+3$ involvit, -1 est, ut constat, non — residuum quadrat ipsius A, unde aequatio $(\frac{1}{2}\vartheta_2)^2 - (\frac{1}{2}\vartheta_1)^2 A = -1$ locum habere nequit, ideoque

$\alpha\alpha$) ϱ_2 nunquam unitas erit atque

$\beta\beta$) multitudo periodi terminorum par.

c) Superest, ut casum disquiramus, in quo numerus A nullum factorem primum $4m+3$ involvat. Multitudinem periodi terminorum tum parem esse posse tum imparem, compluribus exemplis illustratur, quorum haec duo afferri liceat.

Exempl. 1. $A = 13.\ 17 = 221$

$$\sqrt{221} = 14 + \frac{\sqrt{221}-14}{1}$$

$$\frac{1}{\sqrt{221}-14} = \frac{\sqrt{221}+14}{25} = 1 + \frac{\sqrt{221}-11}{25}$$

$$\frac{25}{\sqrt{221}-11} = \frac{\sqrt{221}+11}{4} = 6 + \frac{\sqrt{221}-13}{4}$$

$$\frac{4}{\sqrt{221}-13} = \frac{\sqrt{221}+13}{13} = 2 + \frac{\sqrt{221}-13}{13}$$

$$\frac{13}{\sqrt{221}-13} = \frac{\sqrt{221}+13}{4} = 6 + \frac{\sqrt{221}-11}{4}$$

$$\frac{4}{\sqrt{221}-11} = \frac{\sqrt{221}+11}{25} = 1 + \frac{\sqrt{221}-14}{25}$$

$$\frac{25}{\sqrt{221}-14} = \frac{\sqrt{221}+14}{1} = 28 + \frac{\sqrt{221}-14}{1}$$

Multitudo k in exemplo proposito est 6, i. e. par.

Exempl. 2. $A = 5^5 . 13 = 325$

$$\sqrt{325} = 18 + \frac{\sqrt{325}-18}{1}$$

$$\frac{1}{\sqrt{325}-18} = \frac{\sqrt{325}+18}{1} = 36 + \frac{\sqrt{325}-18}{1}$$

In hoc exemplo numerus k est 1, i. e. impar.

*) Théorie des nombres. Tom. I. p. 71:
Cette conclusion renferme un des plus beaux théorèmes de la science des nombres savoir „que tout nombre premier $4m+1$ est la somme de deux carrés“ et donne en même temps le moyen de faire cette décomposition d'une manière directe et sans aucun tâtonnement.

Ad decidendum a priori, num multitudo κ par sit an impar, ad aequationem $x^2 - Ay^2 = -1$ refugiendum est, cui si per numeros integros satisfieri potest, κ erit impar, par vero, si ei satisfieri nequit, quo in casu est

$$Mx^2 - Ny^2 = -1,$$

ubi M ab unitate diversus, atque $M = \varrho_2$, $N = \varrho_1$.

Studio licet permulto nondum contigit mihi, ut multitudinem κ in hoc ipso casu a priori cognoscam, tamen non dubito, quin geometrae ingenii acumine valentes totam hanc rem e tenebris, quibus obducta videtur esse, in lucem mox detrahant.

Ceterum e re est, afferre, quod in Sect. II. §. 3. reperimus, periodum semper unum terminum habere, ideoque κ imparem esse, quoties A formae sit a^2+1, qui casus huc pertinet, quum numerus a^2+1 factorem primum $4m+3$ involvere nequeat.

5.

Disquisitio peculiaris formae $A = 4m+3$.

Quoties A formae est $4m+3$, nunquam potest esse $\varrho_2 = 1$, vel $(\frac{1}{2}\vartheta_2)^2 - (\frac{1}{2}\vartheta_1)^2 A = -1$, quoniam -1 non — residuum quadrat. est numeri $4m+3$; unde propositio sequens manat:

„Multitudo terminorum periodi semper par est, quoties numerus A formam $4m+3$ habet."

Scimus porro ex antecedentibus, tum primum, tum secundum systema exstare posse.

Semper autem, si A sit potestas (impar) numeri primi $4m+3$, secundum tantum systema locum habere, inde patet, quod pro systemate primo est $\varrho_1\varrho_2 = A$, quae quidem aequatio exstare nequit, nisi aut $\varrho_1 = 1$ aut $\varrho_2 = 1$; at utrumque falsum est, ergo systema secundum valet, pro quo $\sigma_1\sigma_2^2 = A$. Itaque debet esse aut $\sigma_1 = 1$, $\sigma_2 = A$, ubi $A = 8k+7$ (cf. 2.), aut $\sigma_1 = A$, $\sigma_2 = 1$, ubi $A = 8k+3$.

Unde manant propositiones sequentes:

Quoties A est potestas numeri primi $8k+7$

α) est $\sigma_1 = 1$, $\sigma_2 = A$, ideoque $\vartheta_1^2 - \vartheta_2^2 A = 2$, atque 2 residuum quadraticum potestatis imparis numeri primi $8k+7$,

β) Quoties vero A est protestas numeri primi $8k+3$, est $\sigma_1 = A$, $\sigma_2 = 1$, ideoque $\vartheta_2^2 - \vartheta_1^2 A = -2$, atque -2 residuum quadraticum potestatis imparis numeri primi $8k+3$.

6.

Demonstratio unius partis theorematis fundamentalis ad doctrinam numerorum spectantis.

Si numerus A productum est duorum numerorum primorum M, N, qui ambo sunt formae $4m+1$, ex praecedentibus aequatio exstat $(\frac{1}{2}\vartheta_1)^2\varrho_1 - (\frac{1}{2}\vartheta_2)^2\varrho_2 = 1$, ubi $\varrho_1\varrho_2 = MN = A$.

Quum jam nec ϱ_1, nec ϱ_2 unitas esse possit, erit aut $\varrho_1 = M$, $\varrho_2 = N$, aut $\varrho_1 = N$, $\varrho_2 = M$, unde aequationum $(\frac{1}{2}\vartheta_1)^2 M - (\frac{1}{2}\vartheta_2)^2 N = 1$, $(\frac{1}{2}\vartheta_1)^2 N - (\frac{1}{2}\vartheta_2)^2 M = 1$ aut una aut altera locum habere debet.

Si prima locum habet, manifesto M est residuum quadr. ipsius N; quumque tum $(\frac{1}{2}\vartheta_2)^2 N - (\frac{1}{2}\vartheta_1)^2 M = -1$, atque -1 sit non — residuum ipsius M, erit N non — residuum ipsius M.

Sin aequatio secunda locum habet, manifesto N residuum quadr. ipsius M, tumque M non — residuum ipsius N.

Itaque si numeri M, N ambo ejusdem formae $4m + 3$ sunt, alter erit residuum quadraticum alterius, et hic ipse non — residuum quadraticum illius. *)

7.

Connexus numerorum praecedentium cum fractionis continuae elementis, cujus periodus parem terminorum multitudinem habeat.

Vidimus in praecedentibus, multitudinem $\varkappa$ parem esse, quoties A factorem primum $4m + 3$ involvat, atque parem esse posse, si A nullum factorem primum illius formae habeat.

Accepto igitur A ita, ut $\varkappa$ sit par, consideremus fractionem continuam

$$\frac{p_{k-1}}{q_{k-1}} = a + \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_{\frac{1}{2}k}} + \frac{1}{a_{\frac{1}{2}k-1}} + \ldots + \frac{1}{a_1}.$$

Valor ejus per fractiones convergentes $\frac{p_{\frac{1}{2}k-1}}{q_{\frac{1}{2}k-1}}$, $\frac{p_{\frac{1}{2}k-2}}{q_{\frac{1}{2}k-2}}$ hoc modo exprimitur:

Quum facile reperiatur

$$\frac{1}{a_{\frac{1}{2}k-1}} + \frac{1}{a_{\frac{1}{2}k-2}} + \ldots + \frac{1}{a_1} = \frac{q_{\frac{1}{2}k-2}}{q_{\frac{1}{2}k-1}},$$

habetur, ut notum est

$$\frac{p_{k-1}}{q_{k-1}} = \frac{p_{\frac{1}{2}k-1}\left\{a_{\frac{1}{2}k} + \frac{q_{\frac{1}{2}k-2}}{q_{\frac{1}{2}k-1}}\right\} + p_{\frac{1}{2}k-2}}{q_{\frac{1}{2}k-1}\left\{a_{\frac{1}{2}k} + \frac{q_{\frac{1}{2}k-2}}{q_{\frac{1}{2}k-1}}\right\} + q_{\frac{1}{2}k-2}}$$

vel factis nonnullis reductionibus, si brev. gratia ponimus

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*) Restat casus, in quo alter numerorum primorum formam $4m + 1$ habet. Tum quidem M esse residuum vel non — residuum alterius N, prout N sit residuum vel non — residuum ipsius M, ex altera parte theorematis fundamentalis patet, quod ab Ill. Gaussio compluribus modis demonstratum est.

Totius theorematis fundamentalis Gaussius sex demonstrationes ingenio suo indagavit, quarum duae expositae sunt in Disq. Arithm. (Sect. IV. et V.), tertia in commentatione peculiari (Comm. Soc. reg. Gott. Vol. XVI.), quarta in commentatione „Summatio quarundam serierum singularium“ (Comm. recent. Vol. I.), quinta et sexta in commentatione „Theorematis fundamentalis in doctrina de residuis quadraticis demonstrationes et ampliationes novae. Gottingae 1818.“

$$1.\quad G = a_{\frac{1}{2}k}\, q_{\frac{1}{2}k-1} + 2\, q_{\frac{1}{2}k-2}.$$

$$2.\quad \begin{cases} p_{k-1} + (-)^{\frac{1}{2}k} = p_{\frac{1}{2}k-1}\, G \\ q_{k-1} = q_{\frac{1}{2}k-1}\, G \end{cases}$$

Quum $p_{\frac{1}{2}k-1}$ et $q_{\frac{1}{2}k-1}$ inter se primi sint, numerus G divisor comm. maximus erit numerorum $p_{k-1} + (-1)^{\frac{1}{2}k}$, q_{k-1}, unde $G = \vartheta_1$, si $\frac{1}{2}k$ par, at $G = \vartheta_2$, si $\frac{1}{2}k$ impar.

Jam est

$$3.\quad p_{\frac{1}{2}k-1}^2 - A\, q_{\frac{1}{2}k-1}^2 = (-1)^{\frac{1}{2}k} D_{\frac{1}{2}k},$$

ubi $D_{\frac{1}{2}k}$ denominator quotientis medii completi, ergo ex aequationibus 2.

$$4.\quad 2\, p_{\frac{1}{2}k-1} = D_{\frac{1}{2}k} \,.\, G$$

(A.) Quodsi $D_{\frac{1}{2}k}$ impar est, ob relat. 4. numerum $p_{\frac{1}{2}k-1}$ metiri debet, ergo ob relat. 3. etiam A, unde

$$5.\quad A = D_{\frac{1}{2}k} A'.$$

Hinc relat. 4. in hanc mutatur

$$6.\quad D_{\frac{1}{2}k} . \left(\frac{G}{2}\right)^2 - A' . q_{\frac{1}{2}k-1}^2 = (-1)^{\frac{1}{2}k},$$

unde manat, numeros $D_{\frac{1}{2}k}$, A' esse impares inter seque primos.

Porro prima relatt. 2., si pro $p_{\frac{1}{2}k-1}$ valor $\frac{1}{2} D_{\frac{1}{2}k} G$ ex rel. 5. accipiatur, in hanc mutatur $\frac{1}{2}\{p_{k-1} + (-1)^{\frac{1}{2}k}\} = D_{\frac{1}{2}k} . \left(\frac{G}{2}\right)^2$, unde $\frac{1}{2}\{p_{k-1} - (-1)^{\frac{1}{2}k}\} = A' q_{\frac{1}{2}k-1}^2$. Ex quo sequitur, numerum $2\, q_{\frac{1}{2}k-1}$ divisorem comm. max. esse numerorum $p_{k-1} - (-1)^{\frac{1}{2}k}$, q_{k-1}. Posito igitur

$$7.\quad 2\, q_{\frac{1}{2}k-1} = H,$$

erit

$$8.\quad D_{\frac{1}{2}k} . \left(\frac{G}{2}\right)^2 - A' . \left(\frac{H}{2}\right)^2 = (-1)^{\frac{1}{2}k}.$$

(B.) Si $D_{\frac{1}{2}k}$ par est, habetur $p_{\frac{1}{2}k-1} = \frac{1}{2} D_{\frac{1}{2}k} . G$, unde $\frac{1}{2} D_{\frac{1}{2}k}$ metitur numerum $p_{\frac{1}{2}k-1}$, ergo etiam ob rel. 3. numerum A.

Fit igitur ponendo

$$9.\quad A = \tfrac{1}{2} D_{\frac{1}{2}k} A'.$$

$$10.\quad \tfrac{1}{2} D_{\frac{1}{2}k} . G^2 - A' . q_{\frac{1}{2}k-1}^2 = (-1)^{\frac{1}{2}k} 2.$$

Porro est ex relat. 2: $p_{k-1} + (-1)^{\frac{1}{2}k} = \frac{1}{2} D_{\frac{1}{2}k} . q^2$, unde $p_{k-1} - (-1)^{\frac{1}{2}k} = A' . q_{\frac{1}{2}k-1}^2$, ideoque $q_{\frac{1}{2}k-1}$ divisor comm. max. numerorum $p_{k-1} - (-1)^{\frac{1}{2}k}$, q_{k-1}. Ponendo igitur

$$11.\quad q_{\frac{1}{2}k-1} = H$$

fit

$$12.\quad \tfrac{1}{2} D_{\frac{1}{2}k} . G^2 - A' H^2 = (-1)^{\frac{1}{2}k} 2.$$

Summa harum disquisitionum haec est:

(A.) Quoties denominator quotientis medii completi impar est, habetur

$$\left.\begin{aligned} G &= \vartheta_1 \\ D_{\frac{1}{2}k} &= \varrho_1 \\ 2\, q_{\frac{1}{2}k-1} &= \vartheta_2 \\ A' &= \varrho_2 \end{aligned}\right\} \text{pro pari } \tfrac{1}{2}k. \qquad \text{at} \qquad \left.\begin{aligned} G &= \vartheta_2 \\ D_{\frac{1}{2}k} &= \varrho_2 \\ 2\, q_{\frac{1}{2}k-1} &= \vartheta_1 \\ A' &= \varrho_1 \end{aligned}\right\} \text{pro impari } \tfrac{1}{2}k.$$

(B.) Quoties vero denominator quot. med. compl. par est, habetur

$$\left.\begin{aligned} G &= \vartheta_1 \\ \tfrac{1}{2} D_{\frac{1}{2}k} &= \varrho_1 \\ q_{\frac{1}{2}k-1} &= \vartheta_2 \\ A' &= \varrho_2 \end{aligned}\right\} \text{pro pari } \tfrac{1}{2}k. \quad \text{at} \quad \left.\begin{aligned} G &= \vartheta_2 \\ \tfrac{1}{2} D_{\frac{1}{2}k} &= \varrho_2 \\ q_{\frac{1}{2}k-1} &= \vartheta_1 \\ A' &= \varrho_1 \end{aligned}\right\} \text{pro impari } \tfrac{1}{2}k.$$

Unde propositio sequitur:

Si numerus A formam $4m+1$ habet atque multitudo periodi terminorum par est, denominator quotientis medii completi semper impar erit.

8.

Numeri p_{k-1}, q_{k-1} ex praeced. formulis facillime ita determinantur:

Evolvatur fractio continua usque ad quotientem medium completum ponaturque

$$G = a_{\frac{1}{2}k}\, q_{\frac{1}{2}k-1} + 2\, q_{\frac{1}{2}k-2}.$$

(A.) Quodsi denominator quotientis medii completi impar est, ad numeros p_{k-1}, q_{k-1} determinandos relationes habentur

$$\tfrac{1}{2}\left(p_{k-1} + (-1)^{\frac{1}{2}k}\right) = D_{\frac{1}{2}k} \cdot \left(\frac{G}{2}\right)^2$$

$$q_{k-1} = q_{\frac{1}{2}k-1} G.$$

(B.) Sin denominator quotientis medii completi par est, ad numeros p_{k-1}, q_{k-1} determinandos relationes habentur

$$p_{k-1} + (-1)^{\frac{1}{2}k} = \tfrac{1}{2} D_{\frac{1}{2}k} \cdot G^2$$

$$q_{k-1} = q_{\frac{1}{2}k-1} \cdot G.$$

Itaque numeri p_{k-1}, q_{k-1} ab solis numeris quattuor pendent $a_{\frac{1}{2}k}$, $B_{\frac{1}{2}k}$, $q_{\frac{1}{2}k-1}$, $q_{\frac{1}{2}k-2}$.

9.

Disquisitio peculiaris elementorum fractionis continuae in medio positorum.

Ex relationibus notis $J_n + J_{n+1} = a_n D_n$, $D_n D_{n+1} = A - J_{n+1}^2$, ubi $\frac{\sqrt{A} + J_n}{D_n}$ quotiens aliquis completus, pro $n = \frac{1}{2}k$ hae manant

$$13. \quad 2J_{\frac{1}{2}k} = a_{\frac{1}{2}k} D_{\frac{1}{2}k}.$$

$$14. \quad D_{\frac{1}{2}k} D_{\frac{1}{2}k+1} = A - J_{\frac{1}{2}k}^2.$$

(A.) Quodsi $D_{\frac{1}{2}k}$ impar est, erit $a_{\frac{1}{2}k}$ par atque $J_{\frac{1}{2}k} = \frac{1}{2} a_{\frac{1}{2}k} . D_{\frac{1}{2}k}$. Quo valore in aequatione 14. substituto nascitur $D_{\frac{1}{2}k} D_{\frac{1}{2}k+1} = A - (\frac{1}{2} a_{\frac{1}{2}k})^2 D_{\frac{1}{2}k}^2$, unde A per $D_{\frac{1}{2}k}$ divisibilis est.

Ponendo igitur, ut antea, $A = D_{\frac{1}{2}k} A'$, aequatio oritur

$$15. \quad D_{\frac{1}{2}k+1} = A' - (\tfrac{1}{2} a_{\frac{1}{2}k})^2 D_{\frac{1}{2}k},$$

ex qua est

$$16. \quad A' \; 7 \; D_{\frac{1}{2}k}.$$

Si designamus porro per $G(z)$ maximum numerum integrum quantitate z comprehensum, erit $G\left(\frac{\sqrt{A}+J_{\frac{1}{2}k}}{D_{\frac{1}{2}k}}\right) = a_{\frac{1}{2}k}$, ideo $\frac{\sqrt{A}+J_{\frac{1}{2}k}}{D_{\frac{1}{2}k}} \; 7 \; a_{\frac{1}{2}k}$ et $\angle \; a_{\frac{1}{2}k}+1$, vel $\sqrt{A}+J_{\frac{1}{2}k} \; 7 \; a_{\frac{1}{2}k} D_{\frac{1}{2}k}$ et $\angle \; a_{\frac{1}{2}k} D_{\frac{1}{2}k} + D_{\frac{1}{2}k}$, unde

$$17. \quad \sqrt{A} \; 7 \; J_{\frac{1}{2}k} \text{ et } \angle \; J_{\frac{1}{2}k} + D_{\frac{1}{2}k}.$$

(B.) Sin $D_{\frac{1}{2}k}$ par est, habetur $J_{\frac{1}{2}k} = a_{\frac{1}{2}k} \cdot \frac{1}{2} D_{\frac{1}{2}k}$, unde $D_{\frac{1}{2}k} D_{\frac{1}{2}k+1} = A - a_{\frac{1}{2}k}{}^2 . (\frac{1}{2} D_{\frac{1}{2}k})^2$, vel $2 D_{\frac{1}{2}k+1} \cdot \frac{1}{2} D_{\frac{1}{2}k} = A - a_{\frac{1}{2}k}{}^2 . (\frac{1}{2} D_{\frac{1}{2}k})^2$; ergo $\frac{1}{2} D_{\frac{1}{2}k}$ metitur numerum A, unde, si ponimus $A = \frac{1}{2} D_{\frac{1}{2}k} A'$:

$$18. \quad 2 D_{\frac{1}{2}k+1} = A' - a_{\frac{1}{2}k}{}^2 . \tfrac{1}{2} D_{\frac{1}{2}k}.$$

Ex hac aequatione sequitur esse

$$19. \quad A' \; 7 \; \tfrac{1}{2} D_{\frac{1}{2}k}.$$

Porro est $G\left(\frac{\sqrt{A}+J_{\frac{1}{2}k}}{D_{\frac{1}{2}k}}\right) = a_{\frac{1}{2}k}$, ergo $\frac{\sqrt{A}+J_{\frac{1}{2}k}}{D_{\frac{1}{2}k}} \; 7 \; a_{\frac{1}{2}k}$ et $\angle \; a_{\frac{1}{2}k} + 1$, vel $\sqrt{A}+J_{\frac{1}{2}k} \; 7 \; a_{\frac{1}{2}k} D_{\frac{1}{2}k}$ et $\angle \; a_{\frac{1}{2}k} D_{\frac{1}{2}k} + D_{\frac{1}{2}k}$, ideoque ut antea

$$20. \quad \sqrt{A} \; 7 \; J_{\frac{1}{2}k} \text{ et } \angle \; J_{\frac{1}{2}k} + D_{\frac{1}{2}k}.$$

10.

Casus singulares.

I. Si A est potestas numeri primi $4m+3$, denominator quotientis medii completi impar esse nequit. Nam si esset, haberetur $D_{\frac{1}{2}k} A' = A$, ergo aut $D_{\frac{1}{2}k} = 1$, aut $A' = 1$. Illud fieri nequit ob indolem fractionis continuae; hoc ob relationem $A' \; 7 \; D_{\frac{1}{2}k}$.

Itaque denominator, quem dixi, par est, ex quo $A = \frac{1}{2} D_{\frac{1}{2}k} A'$, ergo $\frac{1}{2} D_{\frac{1}{2}k} = 1$, vel $D_{\frac{1}{2}k} = 2$. Porro est (17.) $J_{\frac{1}{2}k} \angle \sqrt{A}$ et $7 \; \sqrt{A} - 2$, unde $J_{\frac{1}{2}k}$ aut $a - 1$ aut a, designante a maximum integrum radice $\sqrt{A}$ comprehensum. Ex aequatione 18. $a_{\frac{1}{2}k}$ impar est; ergo habemus

$$21. \quad \begin{cases} D_{\frac{1}{2}k} = 2. \\ J_{\frac{1}{2}k} \text{ (impar)} = a - 1 \text{ vel } a. \\ a_{\frac{1}{2}k} \text{ (impar)} = J_{\frac{1}{2}k} = a - 1 \text{ vel } a. \end{cases}$$

II. Si A est productum duarum potestatum numerorum primorum vel $A = U^u V^v$, ubi $U^u \angle V^v$, atque

(A.) denominator quot. med. completi impar, erit $D_{\frac{1}{2}k} A' = U^u V^v$, ergo, quum sit $A' \; 7 \; D_{\frac{1}{2}k}$, $D_{\frac{1}{2}k} = U^u$ atque $A' = V^v$.

Porro est ex rel. 17. $J_{\frac{1}{2}k} \angle \sqrt{A}$ et $7 \; \sqrt{A} - U^u$, i. e. $\angle \; U^{\frac{1}{2}u} V^{\frac{1}{2}v}$ et $7 \; U^{\frac{1}{2}u} V^{\frac{1}{2}v} - U^u$, vel

$$22. \quad J_{\frac{1}{2}k} \angle \; U^{\frac{1}{2}u} V^{\frac{1}{2}v} \text{ et } 7 \; U^{\frac{1}{2}u} (V^{\frac{1}{2}v} - U^{\frac{1}{2}u}).$$

Numerus $a_{\frac{1}{2}k}$ determinatur relat. 13.

(B.) Si vero $D_{\frac{1}{2}k}$ par est, habetur $\frac{1}{2} D_{\frac{1}{2}k} A' = U^u V^v$, ergo $\frac{1}{2} D_{\frac{1}{2}k} = U^u$.

Porro est $J_{\frac{1}{2}k} \angle \sqrt{A}$ et $7 \; \sqrt{A} - 2U^u$, i. e. $\angle \; U^{\frac{1}{2}u} V^{\frac{1}{2}v}$ et $7 \; U^{\frac{1}{2}u} V^{\frac{1}{2}v} - 2U^u$, vel

$$23. \quad J_{\frac{1}{2}k} \angle \; U^{\frac{1}{2}u} V^{\frac{1}{2}v} \text{ et } 7 \; U^{\frac{1}{2}u} V^{\frac{1}{2}v} - 2U^{\frac{1}{2}u}.$$

II. Inquiratur in aequationem
$$p^2 - A q^2 = 1,$$
ubi A potestas impar numeri 2 est.

Quando $p^2 - 2^n q^2 = 1$, vel $(p+1)(p-1) = 2^n q^2$, numerus p impar esse debet, qua re aequatio nostra in hanc mutari potest $\frac{p+1}{2} \cdot \frac{p-1}{2} = 2^{n-2} q^2$.

Quodsi ϑ divisor comm. max. est numerorum $\frac{p+1}{2}$, q, vel $\frac{p+1}{2} = \vartheta^2 \varrho_1$, $q = \vartheta q'$, habetur $\varrho_1 \cdot \frac{p-1}{2} = 2^{n-2} q'^2$. Atqui q' ad ϱ_1 primus est, ergo q'^2 metitur $\frac{p-1}{2}$, vel est $\frac{p-1}{2} = q'^2 \varrho_2$, unde $\varrho_1 \varrho_2 = 2^{n-2}$, atque $\varrho_1 = 2^\lambda$, $\varrho_2 = 2^\mu$, ubi $\lambda + \mu = n-2$. Quum jam sit $\frac{p+1}{2} - \frac{p-1}{2} = \vartheta^2 \varrho_1 - q'^2 \varrho_2$ vel $\vartheta^2 \varrho_1 - q'^2 \varrho_2 = 1$, erit $\vartheta^2 2^\lambda - q'^2 2^\mu = 1$, unde aut $\lambda = 0$, aut $\mu = 0$. In casu posteriore est $q'^2 - 2^{n-2} \vartheta^2 = -1$; atqui $q'^2 + 1$ formam habet $8k+2$, ergo $q'^2 + 1$ per 2^{n-2} divisibilis esse nequit, nisi $n \gtreqless 3$.

Si igitur est $n > 3$, habetur

$$1. \quad \begin{cases} \frac{1}{2}(p+1) = \vartheta^2 \\ \frac{1}{2}(p-1) = \left(\frac{q}{\vartheta}\right)^2 \cdot 2^{n-2} \\ \vartheta^2 - 2^{n-2} \cdot \left(\frac{q}{\vartheta}\right)^2 = 1 \end{cases}$$

Ponendo $\vartheta = p_0$, $\frac{q}{\vartheta} = q_0$ fit

$$2. \quad \begin{cases} p = 2 p_0^2 - 1 \\ q = p_0 q_0 \\ p_0^2 - 2^{n-2} q_0^2 = 1. \end{cases}$$

Ceterum p_0, q_0 sunt minimi numeri aequationi $x^2 - 2^{n-2} y^2 = 1$ satisfacientes. Nam si minores exstarent t, u, numeri $2t^2 - 1$, tu aequationi satisfacerent $x'^2 - 2^n y'^2 = 1$, quumque manifesto sit $2t^2 - 1 < p$, numeri p, q non essent minimi aequationi $x'^2 - 2^n y'^2$ satisfacientes.

Quum jam valores p_0, q_0 pro radice $\sqrt{8}$ reperiri possint, minimi valores aequationis $p^2 - 2^n q^2 = 1$, sine ulla evolutione radicis $\sqrt{2^n}$ in fractionem continuam indagari poterunt.

Est enim
$$\sqrt{8} = 2 + \frac{\sqrt{8} - 2}{1}$$
$$\frac{1}{\sqrt{8} - 2} = \frac{\sqrt{8} + 2}{4} = 1 + \frac{\sqrt{8} - 2}{4}$$
$$\frac{4}{\sqrt{8} - 2} = \frac{\sqrt{8} + 2}{1} = 4 + \frac{\sqrt{8} - 2}{1}$$

ergo $p_0 = 3$, $q_0 = 1$.

Itaque ex relatt. 2. habentur successive radices aequationis $p^2 - 2^n q^2 = 1$, nempe

pro $n = 3$,	$p = 3$,	$q = 1$.
„ $n = 5$,	$p = 17$,	$q = 3$.
„ $n = 7$,	$p = 577$,	$q = 51$.
„ $n = 9$,	$p = 665857$,	$q = 29427$.
	etc.	etc.

Quum aequatio $p^2 - 2^n q^2 = -1$ resolvi nequeat, quoties $n > 1$, sequitur „multitudinem periodi terminorum fractionis continuae radicis $\sqrt{2^n}$ parem esse, excepto casu, in quo $n = 1$.

Exempl. $A = 2^5 = 32$.

$$\sqrt{32} = 5 + \frac{\sqrt{32} - 5}{1}$$

$$\frac{1}{\sqrt{32} - 5} = \frac{\sqrt{32} + 5}{7} = 1 + \frac{\sqrt{32} - 2}{7}$$

$$\frac{7}{\sqrt{32} - 2} = \frac{\sqrt{32} + 2}{4} = 1 + \frac{\sqrt{32} - 2}{4}$$

$$\frac{4}{\sqrt{32} - 2} = \frac{\sqrt{32} + 2}{7} = 1 + \frac{\sqrt{32} - 5}{7}$$

$$\frac{7}{\sqrt{32} - 5} = \frac{\sqrt{32} + 5}{1} = 10 + \frac{\sqrt{32} - 5}{1}$$

In hoc exemplo multitudo κ est 4.

III. Inquiratur in aequationem

$$p^2 - Aq^2 = 1,$$

ubi A productum potestatis 2^n et numeri imparis A'.

Si aequatio habetur $p^2 - 2^n A' q^2 = 1$, vel $(p+1)(p-1) = 2^n A' q^2$, numerus p impar erit, quare aequatio in hanc mutari potest $\frac{p+1}{2} \cdot \frac{p-1}{2} = 2^{n-2} A' q^2$.

Quodsi est ϑ divisor comm. maximus numerorum $\frac{p+1}{2}$, q, vel $\frac{q+1}{2} = \vartheta^2 \varrho_1$, $q = \vartheta q'$, erit $\varrho_1 . \frac{p-1}{2} = 2^{n-2} A' q'^2$. Atqui q' ad ϱ_1 primus est, ergo $\frac{p-1}{2} = q'^2 \varrho_2$ ideoque $\varrho_1 \varrho_2 = 2^{n-2} A'$.

Simul relationem habemus $\vartheta^2 \varrho_1 - \left(\frac{q}{\vartheta}\right)^2 \varrho_2 = 1$ ex qua patet, numeros ϱ_1, ϱ_2, nec non $\vartheta_1, \frac{q}{\vartheta}$ inter se primos esse.

Hinc combinationes sequentes:

$$(1.) \ldots\ldots \varrho_1 = 1, \quad \varrho_2 = 2^{n-2}A', \quad \vartheta^2 - \left(\frac{q}{\vartheta}\right)^2 . 2^{n-2}A' = 1$$

$$(2.) \ldots\ldots \varrho_2 = 1, \quad \varrho_1 = 2^{n-2}A', \quad \left(\frac{q}{\vartheta}\right)^2 - \vartheta^2 . 2^{n-2}A' = -1$$

$$(3.) \ldots\ldots \varrho_1 = 2^{n-2}, \ \varrho_2 = A', \quad \vartheta^2 . 2^{n-2} - \left(\frac{q}{\vartheta}\right)^2 A' = 1$$

$$(4.) \ldots\ldots \varrho_2 = 2^{n-2}, \ \varrho_1 = A', \quad \left(\frac{q}{\vartheta}\right)^2 . 2^{n-2} - \vartheta^2 A' = -1.$$

Quodsi accipiamus esse $n > 3$, aequatio (2) locum habere nequit, quia $\left(\frac{q}{\vartheta}\right)^2 + 1$ per 2^{n-2} divisibilis esse nequit. Deinde quum $1 + \left(\frac{q}{\vartheta}\right)^2 A'$ formam $4k+2$ habeat, quoties A' formae est $4m+1$, tum etiam aequatio (3) exstare nequit. Postremo quum $-1 + \vartheta^2 A'$ formam $4k+2$ habeat, quoties A' formae est $4m+3$, tum etiam aequatio (4) locum habere nequit.

Unde propositiones sequentes:

1) Si A' formam $4m+1$ habet, una harum aequationum exstabit

$$\vartheta^2 - \left(\frac{q}{\vartheta}\right)^2 . 2^{n-2}A' = 1,$$

$$\left(\frac{q}{\vartheta}\right)^2 . 2^{n-2} - \vartheta^2 A' = -1;$$

2) si vero A' formae est $4m+3$, una harum

$$\vartheta^2 - \left(\frac{q}{\vartheta}\right)^2 . 2^{n-2}A' = 1,$$

$$\vartheta^2 . 2^{n-2} - \left(\frac{q}{\vartheta}\right)^2 A' = 1.$$

Veniamus nunc ad indolem numeri κ.

Quia $p^2 + 1$ utpote formae $8k+2$ per 2^n divisibilis esse nequit, quoties $n > 1$, in hoc casu aequatio $p^2 - 2^n A' q^2 = -1$ resolvi nequit, qua ex re „multitudo periodi terminorum fractionis continuae evolutione radicis $\sqrt{A}$ exortae semper par erit, quoties A per 4 divisibilis est."

$$\text{Exempl.} \quad \sqrt{28} = 5 + \frac{\sqrt{28}-5}{1}$$

$$\frac{1}{\sqrt{28}-5} = \frac{\sqrt{28}+5}{3} = 3 + \frac{\sqrt{28}-4}{3}$$

$$\frac{3}{\sqrt{28}-4} = \frac{\sqrt{28}+4}{4} = 2 + \frac{\sqrt{28}-4}{4}$$

$$\frac{4}{\sqrt{28}-4} = \frac{\sqrt{28}+4}{3} = 3 + \frac{\sqrt{28}-5}{3}$$

$$\frac{3}{\sqrt{28}-5} = \frac{\sqrt{28}+5}{1} = 10 + \frac{\sqrt{28}-5}{1}$$

Hic est $\kappa = 4$.

Si porro $n = 1$, est $\frac{p^2+1}{2} = \frac{8k+2}{2} = A'q^2$ vel $4k+1 = A'q^2$. Quum jam, si A' formae $4m+3$, $A'q^2$ sit ejusdem formae, aequatio $p^2 - 2A'q^2 = -1$ in hoc casu exstare nequit, unde

„multitudo k semper par erit, quoties A duplum numeri im-
„paris forma $4m + 3$ praediti."

Exempl. $\sqrt{14} = 3 + \frac{\sqrt{14}-3}{1}$

$$\frac{1}{\sqrt{14}-3} = \frac{\sqrt{14}+3}{5} = 1 + \frac{\sqrt{14}-2}{5}$$

$$\frac{5}{\sqrt{14}-2} = \frac{\sqrt{14}+2}{2} = 2 + \frac{\sqrt{14}-2}{2}$$

$$\frac{2}{\sqrt{14}-2} = \frac{\sqrt{14}+2}{5} = 1 + \frac{\sqrt{14}-3}{5}$$

$$\frac{5}{\sqrt{14}-3} = \frac{\sqrt{14}+3}{1} = 6 + \frac{\sqrt{14}-3}{1}$$

Hic est $k = 4$.

Si denique A est duplum numeri imparis forma $4m+1$ praediti, multitudo k tum par, tum impar esse potest, ut exempla haec docent:

$$\sqrt{10} = 3 + \frac{\sqrt{10}-3}{1}$$

$$\frac{1}{\sqrt{10}-3} = \frac{\sqrt{10}+3}{1} = 6 + \frac{\sqrt{10}-3}{1}$$

ubi $k = 1$.

$$\sqrt{18} = 4 + \frac{\sqrt{18}-4}{1}$$

$$\frac{1}{\sqrt{18}-4} = \frac{\sqrt{18}+4}{2} = 4 + \frac{\sqrt{18}-4}{2}$$

$$\frac{2}{\sqrt{18}-4} = \frac{\sqrt{18}+4}{1} = 8 + \frac{\sqrt{18}-4}{1}$$

ubi $k = 2$.

Sectio V.

De convergentia fractionum continuarum.

1.

Convergentiae fractionum continuarum doctrinae, etiamsi vim haud minorem in omnes matheseos partes exerceat, quam doctrina convergentiae serierum infinitarum, tamen duo modo geometrae, quod sciam, Ill. Grunertus *) et Schlömilchius**) operam navarunt.

Pro argumenti hujus gravitate consilium cepi, ut quod illi viri in lucem protulerint, accuratius perpendam atque examinem. Cui rei ut operam darem ideo praesertim impellebar, quod quae Grunertus et Schlömilchius eruerint, inter se discrepant.

Praeterae Gruneri disquisitionibus innixus theoriam convergentiae fractionum continuarum ulterius promovi.

Agetur tamen de iis modo fractionibus continuis, in quibus omnes termini positivi nec non numerator et denominator termini cujusvis signo positivo praediti sunt.

Tales fractiones continuas semper convergentes esse, Ill. Grunertus l. c. argumentatus est.

Sed locum quendam in demonstratione esse, qui vitio aliquo laboret, mox tibi persuasum habebis; ad quod intelligendum ipsam celeberrimi geometrae demonstrationem perlustremus necesse erit.

Designemus fractionem continuam per

$$\text{(f.)}\quad a + \cfrac{\alpha_1}{a_1 + \cfrac{\alpha_2}{a_2 + \cfrac{\alpha_3}{a_3 + \dots \text{ in inf.}}}}$$

Tota res nititur in indole differentiae duarum fractionum $\frac{p_k}{q_k}, \frac{p_{k+1}}{q_{k+1}}$, quam primo accuratius perpendamus.

Ponamus

$$1.\quad \frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k} = \Delta_k.$$

Adjumento relationum notarum $p_{k+1} = p_k a_{k+1} + p_{k-1}\alpha_{k+1}$, $q_{k+1} = q_k a_{k+1} + q_{k-1}\alpha_{k+1}$ sine ulla difficultate relatio reperitur

$$2.\quad \Delta_k = -\Delta_{k-1} \cdot \frac{q_{k-1}\alpha_{k+1}}{q_{k+1}}$$

$$\text{vel } 3.\quad \Delta_k = -\Delta_{k-1} \cdot \left(1 - \frac{q_k a_{k+1}}{q_{k+1}}\right)$$

Ex relatt. 1., 3. manat

$$4.\quad p_{k+1} q_k - p_k q_{k+1} = -(p_k q_{k-1} - p_{k-1} q_k)\,\alpha_{k+1};$$

quum jam sit $p_1 q - p q_1 = \alpha_1$, erit ex rel. 4.

$$5.\quad \Delta_k = (-1)^k \cdot \frac{\alpha_1 \alpha_2 \alpha_3 \dots \alpha_{k+1}}{q_k q_{k+1}}$$

Ex relatt. 3 et 5. sequitur

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*) Beiträge zur reinen und angewandten Mathematik. Erster Theil. III. Brandenburg, 1838.

**) Handbuch der math. Analysis. Erster Theil. Algebr. Analysis. Jena, 1845. pag. 298. sqq.

differentias Δ, Δ_1, Δ_2, Δ_3, etc., alternas positivas ac negativas esse, nempe primam positivam, secundam negativam, tertiam positivam, et sic deinceps, atque eas deinde respectu valorum absolutorum continuo decrescere.

Jam fractio continua (f) converget, quoties valor absolutus ipsius Δ_k, indice in infinitum tendente, ad limitem cifram accedat, quod si non eveniat, fractio continua semper divergens erit.

Habetur enim $\frac{p_1}{q_1} - \frac{p}{q} = \Delta$, $\frac{p_2}{q_2} - \frac{p_1}{q_1} = \Delta_1$, ... $\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k} = \Delta_k$, unde per additionem $\frac{p_{k+1}}{q_{k+1}} = \frac{p}{q} + \Delta + \Delta_1 + \Delta_2 + \dots + \Delta_k$.

Quum autem seriei in dextra parte positae termini inde ab secundo alterni positivi ac negativi sint, sola conditione

$$\text{Lim. } \Delta_k = 0 \text{ pro } k = \infty,$$

ut notum est, conclusio efficitur, fractionem $\frac{p_{k+1}}{q_{k+1}}$ ad limitem finitum convergere. Sin conditio illa deficit, series Δ, Δ_1, Δ_2, etc. nunquam convergens esse potest.

Ad examinandum limitem differentiae Δ_k e re est, relationem 2. disquirere, quae quidem, si $k+1$ pro k ponitur, valorque absolutus ipsius Δ_k per δ_k designatur, in hanc transibit:

$$6. \quad \delta_{k+1} = \delta_k \cdot \frac{q_k \alpha_{k+2}}{q_{k+2}}.$$

Resoluto q_{k+2} in summam $q_{k+1} a_{k+2} + q_k \alpha_{k+2}$ habetur $\delta_{k+1} = \delta_k \cdot \frac{1}{1 + \frac{q_{k+1} a_{k+2}}{q_k \alpha_{k+2}}}$.

Porro est $\frac{q_{k+1} a_{k+2}}{q_k \alpha_{k+2}} = \frac{(q_k a_{k+1} + q_{k-1} \alpha_{k+1}) a_{k+2}}{q_k \alpha_{k+2}} = \frac{a_{k+1} a_{k+2}}{\alpha_{k+2}} + \frac{q_{k-1} \alpha_{k+1} a_{k+2}}{q_k \alpha_{k+2}}$, unde est

$$7. \quad \delta_{k+1} < \delta_k \cdot \frac{1}{1 + \frac{a_{k+1} a_{k+2}}{\alpha_{k+2}}}.$$

Jam ut disquisitio facilior fiat, fractionem cont. (f) ex §. 4. Sect. II. in aliam transformemus, in qua omnes denominatores unitati aequales, scilicet

$$\text{(f.)} \quad a + \cfrac{\alpha_1}{a_1 + \cfrac{\alpha_2}{a_2 + \cfrac{\alpha_3}{a_3 + \cfrac{\alpha_4}{a_4 + \dots}}}} = a + \cfrac{\alpha_1 : a_1}{1 + \cfrac{\alpha_2 : a_1 a_2}{1 + \cfrac{\alpha_3 : a_2 a_3}{1 + \cfrac{\alpha_4 : a_3 a_4}{1 + \text{etc.}}}}}$$

$$= a + \cfrac{z_1}{1 + \cfrac{z_2}{1 + \cfrac{z_3}{1 + \cfrac{z_4}{1 + \text{etc.},}}}}$$

ubi in universum

$$8. \quad \frac{\alpha_{k+1}}{a_k\, a_{k+1}} = z_{k+1}.$$

Hinc fit relat. 7.

$$9. \quad \delta_{k+1} \angle \delta_k . \frac{z_{k+2}}{1+z_{k+2}}.$$

Auctore Grunerto ponamus $\delta_{k+1} = \delta_k . L_{k+2}$, $\delta_{k+2} = \delta_{k+1} . L_{k+3}$, etc. $\delta_{k+p-1} = \delta_{k+p-2} . L_{k+p}$, unde erit $\delta_{k+p-1} = \delta_k . L_{k+2} L_{k+3} \ldots . L_{k+p}$.

Jam limes producti $L_{k+2} L_{k+3} \ldots . L_{k+p}$ examinandus est. Ad hunc finem quod ex conditionibus

$$L_{k+2} \angle \frac{z_{k+2}}{1+z_{k+2}},\; L_{k+3} \angle \frac{z_{k+3}}{1+z_{k+3}},\; \text{etc.}$$

concludi possit videamus.

Primum perspicuum erit, fractionem $\frac{z_{k+p}}{1+z_{k+p}}$ eo majorem fore, quo major sit z_{k+p}, eamque unitate semper minorem ipsam unitatem limitem habere.

I. Quoties z_{k+p}, indice p in infinitum crescente, quantitatem aliquam finitam non superet, una quantitatum $z_{k+2}, z_{k+3}, z_{k+4}, \ldots .$ maxima erit, quam designemus per z. Unde fractionum $\frac{z_{k+2}}{1+z_{k+2}}, \frac{z_{k+3}}{1+z_{k+3}}, \frac{z_{k+4}}{1+z_{k+4}}, \ldots$ maxima $\frac{z}{1+z}$, ideoque productum $\left(\frac{z_{k+2}}{1+z_{k+2}}\right)\left(\frac{z_{k+3}}{1+z_{k+3}}\right) \ldots . \left(\frac{z_{k+p}}{1+z_{k+p}}\right) \angle \left(\frac{z}{1+z}\right)^{p-1}$; potestas autem $\left(\frac{z}{1+z}\right)^{p-1}$ ad limitem cifram convergit, ergo etiam illud productum. Hinc etiam cifra limes est producti $\delta_k . L_{k+2} L_{k+3} \ldots L_{k+p}$ i. e. differentiae δ_{k+p-1}.

Et hactenus quidem nulli dubitationi obnoxium est, quin omnes conclusiones nostrae verae sint. Jam vero vitium demonstrationis, cujus supra mentionem feci, incipit.

II. Si quantitas z_{k+p} simul cum p in infinitum crescit, quo in casu fractio $\frac{z_{k+p}}{1+z_{k+p}} = \frac{1}{1+\frac{1}{z_{k+p}}}$ ad unitatem limitem tendit, ex Grunerti sententia quantitates $L_{k+2}, L_{k+3}, L_{k+4}, \ldots .$ fractionem quandam genuinam x superare nequeunt. Quod falsum est. Etiamsi enim est $L_{k+p} \angle \frac{z_{k+p}}{1+z_{k+p}}$, atque $\frac{z_{k+p}}{1+z_{k+p}}$ ad limitem 1 convergit, nihilominus ipsa quantitas L_{k+p} ad eundem limitem 1 tendere potest, quod si eveniat, quamcunque fractionem genuinam superabit.

Etsi v. c. est $\frac{p-1}{p} \angle \frac{p}{p+1}$, tamen ambae fractiones ad eundem limitem 1 convergunt.*)

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*) Falsa haec Ill. Grunerti conclusio ab indiligente ad notionem infiniti spectante sermone orta esse videtur, cujus culpam insignis noster geometra alias non meret; immo stilus ejus maximae perspicuitatis testimonium praebet.

Jam ex primo casu hoc theorema habemus:

„Fractio continua (f) convergens erit, quoties quantitas

$$\frac{\alpha_{k+1}}{a_k\, a_{k+1}}$$

„k in infinitum crescente, quantitatem aliquam finitam non „superet, vel quod idem valet, si habetur

$$\text{Lim.}\ \frac{a_k\, a_{k+1}}{\alpha_{k+1}} > 0.\text{“}$$

Hoc ipsum theorema Ill. Schlömilchius in opere suo enunciavit egregio quidem sed in eo, ut puto, reprehendendo, quod nusquam auctorum rerum mentio facta sit.

Quum Schlömilchius eandem fere viam, quam Grunertus, ingressus sit, neque hujus geometrae disquisitionum meminerit, eas non videtur cognovisse.

2.

Disquisitio casus, in quo z_{k+p} in infinitum crescit.

Quum ex praecedentibus habeatur

$$\delta_{k+p-1} < \delta_k \cdot \left(\frac{z_{k+2}}{1+z_{k+2}}\right)\left(\frac{z_{k+3}}{1+z_{k+3}}\right)\ldots\ldots,$$

fractio continua convergens erit, quoties productum

$$\left(\frac{z_{k+2}}{1+z_{k+2}}\right)\left(\frac{z_{k+3}}{1+z_{k+3}}\right)\left(\frac{z_{k+4}}{1+z_{k+4}}\right)\ldots,$$

quod ita etiam exhiberi potest

Ceterum in hoc secundo casu disquisitione singulari opus esse, inde patet, quod productum

$$\left(\frac{z_{k+2}}{1+z_{k+2}}\right)\left(\frac{z_{k+3}}{1+z_{k+3}}\right)\left(\frac{z_{k+4}}{1+z_{k+4}}\right)\ldots\ldots\ldots$$

revera ad limitem finitum ab cifra diversum convergere potest. Quod hoc fere exemplo illustratur.

Productum

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\ldots = \frac{3}{4}\cdot\frac{8}{9}\cdot\frac{15}{16}\cdot\frac{24}{25}\ldots\ldots$$

limitem habet $\frac{1}{2}$.

Nam ex nota formula

$$\sin \pi x = \pi x\,(1-x^2)\left(1-\frac{1}{4}x^2\right)\left(1-\frac{1}{9}x^2\right)\ldots\ldots$$

sequitur

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\ldots\ldots = \frac{\sin \pi x}{\pi x(1-x^2)}\ (\text{pro } x = 1).$$

Quum hujus fractionis numerator et denominator pro $x = 1$ evanescant, ambo differentiandi sunt, unde fit

$$\frac{\sin \pi x}{\pi x\,(1-x^2)}\ (\text{pro } x = 1) = \frac{\pi \cos \pi x}{-2\pi x + \pi(1-x^2)}\ (\text{pro } x = 1) = \frac{-\pi}{-2\pi} = \frac{1}{2}.$$

Quum igitur sit

$$\delta_{n+p-1} < \delta_k \cdot \left(\frac{z_{k+2}}{1+z_{k+2}}\right)\left(\frac{z_{k+3}}{1+z_{k+3}}\right)\ldots\ldots,$$

atque dextera pars non semper ad cifram limitem accedat, concludi nequit, δ_{k+p-1} cifram limitem habere.

$$\left(1-\frac{1}{1+z_{k+2}}\right)\left(1-\frac{1}{1+z_{k+3}}\right)\left(1-\frac{1}{1+z_{k+4}}\right)\ldots,$$

ad limitem cifram accedat.

Disquiramus nunc in omni genere productum

$$P=(1+u_0)(1+u_1)(1+u_2)\ldots,$$

ubi quantitates u_0, u_1, u_2, ..., quoties negativae sint, unitatem superare non debent; nam productum convergere nequit, nisi omnes factores positivi sunt.

Sponte quasi hae primae propositiones se praebent:

α) Si factor $1+u_n$ cifram limitem habet, ad hunc ipsum limitem productum P accedet.

β) Si $1+u_n$ ad limitem unitate minorem accedit, etiamtum productum P ad limitem cifram converget.

γ) Si $1+u_n$ in infinitum tendit, ipsum productum in infinitum crescet.

δ) Si $1+u_n$ ad limitem unitate majorem convergit, productum in infinitum crescet.

Superest, ut casum disquiramus, in quo $1+u_n$ ad limitem 1 vel u_n ad limitem cifram accedat.

Tum convergentia producti ad convergentiam serierum reducitur, quando ejus logarithmus accipitur.

Est enim

$$\log. P=\log.(1+u_0)+\log.(1+u_1)+\log.(1+u_2) \text{ etc.}$$

Resolutis logarithmis his in series convergentes, scilicet

$$\log.(1+u_n)=u_n-\tfrac{1}{2}u_n^2+\tfrac{1}{3}u_n^3-\ldots..$$

$$\log.(1+u_{n+1})=u_{n+1}-\tfrac{1}{2}u_{n+1}^2+\tfrac{1}{3}u_{n+1}^3\ldots..$$

$$\log.(1+u_{n+m-1})=u_{n+m-1}-\tfrac{1}{2}u_{n+m-1}^2+\tfrac{1}{3}u_{n+m-1}^3-\ldots..$$

habetur

$$\frac{u_n-\log.(1+u_n)}{u_n^2}=\tfrac{1}{2}-\tfrac{1}{3}u_n+\ldots..$$

$$\frac{u_{n+1}-\log.(1+u_{n+1})}{u_{n+1}^2}=\tfrac{1}{2}-\tfrac{1}{3}u_{n+1}+\ldots..$$

$$\frac{u_{n+m-1}-\log.(1+u_{n+m-1})}{u_{n+m-1}^2}=\tfrac{1}{2}-\tfrac{1}{3}u_{n+m-1}+\ldots..,$$

unde patet, quamque fractionum in sinistra parte positarum ad limitem $\frac{1}{2}$ convergere, si index in infinitum tendat.

Ex theoria quantitatum mediarum (Mittelgrößen) theorema notum est:

„Si a, a′, a″, etc. sunt quantitates quaelibet, b, b′, b″, etc. vero eodem signo praeditae, fractio $\frac{a+a'+a''\text{ etc.}}{b+b'+b''\text{ etc.}}$ quantitas media est inter fractiones singulas $\frac{a}{b}, \frac{a'}{b'}, \frac{a''}{b''}$, etc.“

Quod theorema si ad nostrum casum applicemus, fractio

$$(a.)\quad \frac{\sum\limits_{k=n}^{k=n+m-1} u_k - \sum\limits_{k=n}^{k=n+m-1} \log.\,(1+u_k)}{\sum\limits_{k=n}^{k=n+m-1} u_k^2}$$

quantitas media est inter fractiones

$$\frac{u_n - \log.\,(1+u_n)}{u_n^2},\ \frac{u_{n+1} - \log.\,(1+u_{n+1})}{u_{n+1}^2},\ldots\ \frac{u_{n+m-1} - \log.\,(1+u_{n+m-1})}{u_{n+m-1}^2}.$$

Quae quum limitem $\frac{1}{2}$ habeant, facile patebit, etiam fractionis (a) limitem $\frac{1}{2}$ esse. Unde manat

$$(b.)\quad \text{Lim.} \sum_{k=1}^{k=n+m-1} \log.\,(1+u_k) = \text{Lim.} \sum_{k=1}^{k=n+m-1} u_k - \tfrac{1}{2}\text{Lim.} \sum_{k=1}^{k=n+m-1} u_k^2.$$

Hinc sequentes propositiones habemus:

I. Si series

$$u_0,\ u_1,\ u_2,\ u_3,\ \ldots..\ (A)$$
$$u_0^2,\ u_1^2,\ u_2^2,\ u_3^2,\ldots..\ (B)$$

ambae convergentes sunt, productum

$$(1+u_0)\,(1+u_1)\,(1+u_2)\ \ldots..\ (C)$$

ad limitem finitum ab cifra diversum converget.

II. Si series (A) convergens est, (B) vero divergens, productum (C) cifram limitem habebit.

III. Si series (A) divergens est, (B) vero convergens, productum (C) vel cifram vel infinitum limitem habebit, prout summa seriei (A) est $-\infty$ vel $+\infty$.

IV. Si series (A) convergens est omnesque tandem termini eodem signo praediti sunt, manifesto etiam (B) convergens erit, unde (I.) productum (C) ad limitem finitum ab cifra diversum tendet.

Restat casus, in quo ambae series (A), (B) divergentes sunt. Tum adjumento aequationis (b) de convergentia vel divergentia producti decidi nequit.

Magnae autem utilitatis propositio haec erit:

V. Si series (A) divergens est, omnesque tandem termini eodem signo praediti sunt, productum (C) cifram vel infinitum limitem habebit, prout omnes termini negativi vel positivi fiant.

Demonstratio.

α) Omnes termini fiant positivi.

Fractiones

$$\frac{\log.\,(1+u_n)}{u_n},\ \frac{\log.\,(1+u_{n+1})}{u_{n+1}},\ \ldots..$$

ad limitem 1 tendunt, ut facile per calc. different. patet, ergo etiam quantitas media inter eas

$$\frac{\sum\limits_{k=n}^{k=n+m-1} \log.\,(1+u_k)}{\sum\limits_{k=n}^{k=n+m-1} u_k}$$

ad limitem 1 converget. Cujus fractionis quum denominator pro $n = \infty$ non evanescat, etiam numerator non evanescet, unde $\log. (1 + u_0) + \log. (1 + u_1) + \ldots\ldots = \log. (C) = \infty$, ideoque productum $(C) = \infty$.

β) Casus, in quo omnes termini negativi fiunt, ad primum reducitur.

Applicemus nunc, quae modo eruimus, ad convergentiam producti

$$\left(1 - \frac{1}{1+z_{k+2}}\right) \left(1 - \frac{1}{1+z_{k+3}}\right) \left(1 - \frac{1}{1+z_{k+4}}\right) \ldots\ldots$$

Quum termini u_0, u_1, u_2, ... eodem signo praediti sint, quoties series (A) convergat, etiam (B) convergens erit, ideoque limes producti non cifra (I), unde casus I, et II sunt excludendi.

Itaque ex V. hoc theorema habemus:

Fractio continua (f) convergens erit, quoties quantitas

$$z_{k+1} = \frac{\alpha_{k+1}}{a_k\, a_{k+1}}$$

in infinitum crescat, simulque series

$$\frac{1}{1+z_{k+1}},\ \frac{1}{1+z_{k+2}},\ \frac{1}{1+z_{k+3}},\ \ldots\ldots$$

divergens sit.

Ex hoc theoremate ex. gr. fractio continua convergens est

$$\cfrac{1}{1+\cfrac{2}{1+\cfrac{3}{1+\cfrac{4}{1+\text{ in inf.}}}}}$$

Scribebam Sundiae d. 22. m. Jul. a. MDCCCXLV.